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The Distribution Free Newsboy Problem with Balking

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The purpose of this paper is to study the classic newsboy model with more realistic assumptions. First, we allow customers to balk when inventory level is low. Secondly, we relax the assumption that the cumulative distribution function of the demand is completely known and merely assume that the first two moments of the distribution function are known.

Key words: inventory, newsboy problem

INTRODUCTION

The single period inventory or newsboy model is used to analyse stocking polices for goods with a relatively short shelf-life, e.g. baked goods, seasonal products, and newspaper/periodicals. The trade-off is between the risk of overstocking (forcing disposal below the unit purchasing cost) and the risk of understocking (losing the opportunity of making a profit). The newsboy model is often used in the fashion and the sporting goods industries to aid decision making, both at the manufacturing and at the retail level. See Barron[1] for successful applications of the model to the Hallmark company.

While the classical single-period inventory model aptly describes many products with a short shelf-life, there are some products for which demand is not solely an exogenous variable, but is also a function of inventory position. For example, consider the operation of a Christmas tree lot. It is not unrealistic to assume that when the selection of available trees falls below some threshold, customers desiring a tree may pass up the lot altogether in favour of one with a greater selection. In a similar vein, a major west-coast doughnut company recently began supplying fresh doughnuts on a daily basis to a chain of convenience stores. The doughnuts are placed in specially designed display cases. As the stock in the cases falls below a certain level, management believes that customers tend to balk at purchasing due to a perception that the doughnuts are no longer fresh[2]. In this paper we consider the single-period inventory model with the modification that once the inventory level falls to K or less, the per unit chance of a sale declines from 1 to L. It is clear that the classic single-period inventory problem is a special case of this model. The basic model with balking was initially studied by Pasternack[2], who assumed that the demand distribution is completely known. In this paper we apply the distribution free approach to the basic model of Pasternack[2], and extend this work by considering the fixed ordering cost case.

In addition to the balking behaviour, the distributional information of the demand is very limited in practice. Sometimes all that is available is an educated guess of the mean and of the variance. There is a tendency to use the normal distribution under these conditions. However, the normal distribution does not provide the best protection against the occurrence of other distributions with the same mean and same variance. In a classic paper, Scarf[3] addressed the newsboy problem where only the mean μ and the variance σ^2 of the demand are known without any further assumptions about the form of the distribution of the demand. Taking a conservative approach, he modelled the problem as that of finding the order quantity that maximizes the expected profit against the worst possible distribution of the demand with the mean μ and the variance σ^2. He showed through a beautiful, but lengthy, mathematical argument that the worst distribution of the demand has positive mass at two points and used this result to obtain a closed form expression for the optimal order quantity. The approach is

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called the minmax distribution free approach. Other works related to the distribution free approach include Gallego[4], Moon and Gallego[5] and Shore[6]. The purpose of this paper is to apply the distribution free approach to the newsboy model with balking. This paper is a generalization of the work of Gallego and Moon[7].

BASIC MODEL AND DISTRIBUTION FREE APPROACH

The data for the newsboy problem with balking are as follows:

c = unit cost;
p = unit selling price;
v = unit salvage value;
F = cumulative distribution function of the demand;
μ = expected demand;
σ = standard deviation of the demand;
K = balking level (i.e. inventory level where balking starts);
L = unit chance of a sale when balking occurs;
Q = order quantity.

Let D denote the random demand with probability density function $f(D)$. In what follows we let $x^+ = \max\{x, 0\}$.

The expected profit can be written as

$$\begin{aligned}\pi^F(Q) = &\int_0^{Q-K} [pD + v(Q - D)]f(D)\,\mathrm{d}D \\ &+ \int_{Q-K}^{Q-K+K/L} \{p[Q - K + L(D - Q + K)] + v[K - L(D - Q + K)]\}f(D)\,\mathrm{d}D \\ &+ \int_{Q-K+K/L}^{\infty} pQf(D)\,\mathrm{d}D - cQ. \qquad (1)\end{aligned}$$

In (1) the first term gives the expected profit if demand is between 0 and $Q - K$. Note that balking will not occur for this case since the inventory level does not drop to the balking level K. The second term gives the expected profit if demand is between $Q - K$ and $Q - K + K/L$. Balking will occur for this case, affecting $(1 - L)[D - (Q - K)]$ units among D units demanded. Consequently, we can sell $Q - K + L(D - Q + K)$ units, and $Q - [Q - K + L(D - Q + K)]$ units will not be sold. The third term gives the expected profit if demand is larger than $Q - K + K/L$. Balking will occur for this case, but all items will be sold out since the demand is large enough.

Noting that

$$\int_{Q-K}^{\infty} (D - Q + K)f(D)\,\mathrm{d}D = E[D - Q + K]^+,$$

we can write the expected profit as follows (see the Appendix for the derivation):

$$\begin{aligned}\pi^F(Q) = &(p - v)\mu - (1 - L)(p - v)E[D - Q + K]^+ \\ &- L(p - v)E[D - Q + K - K/L]^+ + (v - c)Q.\end{aligned}$$

Note that if $K = 0$ and $L = 1$, the above equation reduces to the profit function of the classical newsboy problem. Evidently, maximizing $\pi^F(Q)$ is equivalent to minimizing

$$\begin{aligned}C^F(Q) = &(1 - L)(p - v)E[D - Q + K]^+ + L(p - v)E[D - Q + K - K/L]^+ \\ &+ (c - v)Q. \qquad (2)\end{aligned}$$

It is easy to verify that $C^F(Q)$ is strictly convex in Q. Upon setting the derivative to zero, we get

$$(1 - L)F(Q - K) + LF(Q - K + K/L) = \frac{p - c}{p - v}. \qquad (3)$$

Let Q^F be the optimal order quantity when the cumulative distribution of the demand is $F \in \mathcal{F}$. It is easy to find the optimal Q^F satisfying the above equation using a line search.

Now, we consider the distribution free approach. We make no assumption on the distribution F of D other than saying that it belongs to the class $\mathcal{F}$ of cumulative distribution functions with mean μ and variance σ^2. Since the distribution F of D is unknown we want to minimize (2) against the worst possible distribution in $\mathcal{F}$. To this end, we need the following proposition as in Gallego[8].

Proposition 1

$$E[D - Q + K]^+ \leq \frac{(\sigma^2 + (Q - K - \mu)^2)^{1/2} - (Q - K - \mu)}{2} \tag{4}$$

$$E[D - Q + K - K/L]^+ \leq \frac{(\sigma^2 + (Q - K + K/L - \mu)^2)^{1/2} - (Q - K + K/L - \mu)}{2}. \tag{5}$$

Proof

First note that

$$[D - Q + K]^+ = \frac{|D - Q + K| + (D - Q + K)}{2}.$$

Equation (4) follows by taking expectations and using the Cauchy–Schwarz inequality

$$E|D - Q + K| \leq [E(D - Q + K)^2]^{1/2} = [\sigma^2 + (Q - K - \mu)^2]^{1/2}.$$

Equation (5) can be shown similarly.

Remark 1

For the classical newsboy model, Gallego and Moon[7] showed that there exists a distribution in $\mathcal{F}$ where the upper bound is tight for every Q. For the newsboy model with balking, we can similarly prove that there exists a distribution in $\mathcal{F}$ where the upper bound in (4) and (5) is tight, respectively. However, we could not show that there exists a distribution which satisfies the upper bounds simultaneously for each Q.

The distribution free approach for this model is to find the most unfavourable distribution in $\mathcal{F}$ for each Q and then minimize over Q. One reasonable alternative is to minimize the upper bound on $C^F(Q)$. Our problem is now to minimize

$$C^W(Q) = (c - v)Q + (p - v)(1 - L)\frac{[\sigma^2 + (Q - K - \mu)^2]^{1/2} - (Q - K - \mu)}{2}$$
$$+ (p - v)L\frac{[\sigma^2 + (Q - K + K/L - \mu)^2]^{1/2} - (Q - K + K/L - \mu)}{2}.$$

Note that $C^W(Q)$ is strictly convex since

$$\frac{\partial^2 C^W(Q)}{\partial Q^2} = (p - v)\sigma^2\frac{1 - L}{2}[\sigma^2 + (Q - K - \mu)^2]^{-3/2}$$
$$+ (p - v)\sigma^2\frac{L}{2}[\sigma^2 + (Q - K + K/L - \mu)^2]^{-3/2} > 0.$$

Upon using Leibniz's rule and setting the derivative of $C^W(Q)$ to zero, we get

$$(1 - L)\frac{(Q - K - \mu)}{[\sigma^2 + (Q - \mu - K)^2]^{1/2}} + L\frac{(Q - K + K/L - \mu)}{[\sigma^2 + (Q - K + K/L - \mu)^2]^{1/2}} = \frac{(p - c) + (v - c)}{p - v}. \tag{6}$$

Again we can find the optimal Q^W satisfying the above equation using a line search. Here, Q^W is the optimal order quantity against the worst distribution.

Remark 2

If $L = 1$ and $K = 0$, the problem reduces to the classical distribution free newsboy problem, and (6) results in a closed-form solution as in Gallego and Moon[7].

If we use the order quantity Q^W instead of Q^F, the expected loss is equal to

$$\pi^F(Q^F) - \pi^F(Q^W).$$

This is the largest amount that we would be willing to pay for the knowledge of F. This quantity can be regarded as the Expected Value of Additional Information (EVAI).

Example 1

The unit cost is \$35, the unit selling price is \$60, and the unit salvage value is \$15. The mean and the standard deviation of the demand are 800 and 150, respectively. The balking level and the unit chance of a sale are 200 and 0.8, respectively. We compare the performance of Q^W with Q^N where $N \in \mathcal{F}$ represents the normal distribution. The optimal order quantities are $Q^W \simeq 804$ and $Q^N \simeq 829$. The expected profits are $\pi^W(Q^W) = \$16030$ and $\pi^N(Q^N) = \$16780$. The EVAI (calculated with the exact values of Q^W and Q^N) is

$$\pi^N(Q^N) - \pi^N(Q^W) = \$16780.86 - \$16774.72 = \$6.14.$$

Example 2

We use the same data as in Example 1 except that the demand follows a uniform distribution $U(540, 1060)$. Note that the mean and the standard deviation of this distribution are 800 and 150 respectively, which are the same as those in Example 1. We compare the performance of Q^W with Q^U where $U \in \mathcal{F}$ represents the uniform distribution. The optimal order quantities are $Q^W \simeq 804$ and $Q^U \simeq 815$. The expected profits are $\pi^W(Q^W) = \$16030$ and $\pi^U(Q^U) = \$16680$. The EVAI (calculated with the exact values of Q^W and Q^U) is

$$\pi^U(Q^U) - \pi^U(Q^W) = \$16680.24 - \$16652.98 = \$27.26.$$

FIXED ORDERING COST CASE

Let $I \geq 0$ denote the initial inventory and suppose a fixed cost, say A, is charged for placing an order. If $S = I + Q$, then the expected profit can be written as

$$\pi^F(S) = (p - v)\mu - (p - v)\{(1 - L)E[D - S + K]^+ + LE[D - S + K - K/L]^+\} + cI + (v - c)S - A\mathbf{1}_{\{S>I\}},$$

where $\mathbf{1}$ denotes the indicator function.

Using Proposition 1, the problem reduces to

$$\min_{S \geq I}\,[A\mathbf{1}_{\{S>I\}} + G(S)]$$

where

$$G(S) = (1 - L)(p - v)\frac{[\sigma^2 + (S - K - \mu)^2]^{1/2} - (S - K - \mu)}{2} + L(p - v)\frac{[\sigma^2 + (S - K + K/L - \mu)^2]^{1/2} - (S - K + K/L - \mu)}{2} - vS + c(S - I).$$

Let S^* denote the unconstrained minimizer of $G(S)$. From the result of the previous section, we know that

$$(1 - L)\frac{S^* - K - \mu}{[\sigma^2 + (S^* - K - \mu)^2]^{1/2}} + L\frac{S^* - K + K/L - \mu}{[\sigma^2 + (S^* - K + K/L - \mu)^2]^{1/2}} = \frac{(p - c) + (v - c)}{p - v}. \quad (7)$$

By employing a similar argument applied to the classic newsboy model with a fixed ordering cost[9], an order should be placed if $I < S^*$ and $G(I) > A + G(S^*)$. Since $G(S)$ is strictly convex and is not bounded from above, there exists a unique $s^* < S^*$ satisfying

$$G(s^*) = A + G(S^*). \tag{8}$$

The ordering rule is: order up to S^* units if $I < s^*$ and do not order otherwise.

Example 3

We continue Example 1. If there is a fixed ordering cost, say $A = \$500$, then $(s^*, S^*) = (712, 804)$ using (7) and (8). That is, the optimal policy is to order up to 804 units if the initial inventory is less than 712 and not to order otherwise.

COMPUTATIONAL RESULTS

In order to investigate the robustness of the distribution free approach, 1000 test problem instances were randomly generated from uniform distributions on the given intervals. Table 1 shows the distributions for the data set. The mean and the standard deviation are fixed as 800, 150, respectively except for the t distribution.

Table 2 shows the comparative results using several different distributions including normal, uniform, t and triangle distributions. We have reported the minimum, mean, and maximum ratios of $(\pi^F(Q^F))/(\pi^F(Q^W))$ for the 1000 instances. Most of the ratios are quite close to 1, which enables us to use the distribution free ordering rule in the absence of the specific form of the distribution function. It can be seen that the rule works best for the normal distribution. Also, we know from the statistical property that the normal distribution maximizes entropy subject to a fixed mean and variance. We conjecture that there are some kinds of connections between these two facts.

TABLE 1. *Distributions for randomly generated data*

Problem data	p	c	v	K	L
Range	[80, 100]	[40, 60]	[10, 30]	[100, 200]	[0.5, 1]

TABLE 2. *Results of comparative examples*

Distribution	Ratio	Minimum ratio	Mean ratio	Maximum ratio
Normal	$\frac{\pi^N(Q^N)}{\pi^N(Q^W)}$	1.00000	1.00017	1.00118
Uniform	$\frac{\pi^U(Q^U)}{\pi^U(Q^W)}$	1.00000	1.00103	1.00212
t	$\frac{\pi^t(Q^t)}{\pi^t(Q^W)}$	1.00000	1.00032	1.00211
Triangle	$\frac{\pi^{TR}(Q^{TR})}{\pi^{TR}(Q^W)}$	1.00000	1.00022	1.00153

CONCLUDING REMARKS

We have derived the optimal ordering rule for the newsboy problem with balking where only the mean and the variance of the demand are known. We think the distribution free approach is robust, which can be conjectured from numerical examples and computational experiments. Further theoretical investigations on the robustness of the ordering rule resulting from the distribution free approach might be an interesting research problem. Another interesting open problem is to show that there exists a distribution which satisfies the upper bounds in (4) and (5) simultaneously for every Q, as explained in Remark 1.

APPENDIX

Derivation of $\pi^F(Q)$

Each term in (1) can be represented as follows:

$$\int_0^{Q-K}[pD + v(Q-D)]f(D)\,\mathrm{d}D = \int_0^{Q-K}[(p-v)(D-Q+K) + (pQ - pK + vK)]f(D)\,\mathrm{d}D$$
$$= (p-v)(\mu - Q + K) - (p-v)E[D-Q+K]^+$$
$$+ (pQ - pK + vK)F(Q-K). \qquad \text{(A1)}$$

$$\int_{Q-K}^{Q-K+K/L}\{p[Q-K+L(D-Q+K)] + v[K - L(D-Q+K)]\}f(D)\,\mathrm{d}D$$
$$= \int_{Q-K}^{Q-K+K/L} L(p-v)(D-Q+K)f(D)\,\mathrm{d}D + \int_{Q-K}^{Q-K+K/L}[p(Q-K)+vK]f(D)\,\mathrm{d}D$$
$$= L(p-v)E[D-Q+K]^+ - L(p-v)E[D-Q+K-K/L]^+$$
$$+ (pQ - pK + vK)[1 - F(Q-K)] - pQ[1 - F(Q-K+K/L)]. \qquad \text{(A2)}$$

$$\int_{Q-K+K/L}^{\infty} pQf(D)\,\mathrm{d}D - cQ = pQ[1 - F(Q-K+K/L)] - cQ. \qquad \text{(A3)}$$

By adding equations (A1), (A2) and (A3) and simplifying them, we can get $\pi^F(Q)$ as in the text.

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