Int. J. Production Economics 155 (2014) 204–213

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Improved quality, setup cost reduction, and variable backorder costs in an imperfect production process

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ARTICLE INFO

Article history:
Received 29 March 2013
Accepted 13 November 2013
Available online 21 November 2013

Keywords:
Inventory
Quality improvement
Setup cost reduction
Backorder
Distribution free approach

ABSTRACT

This paper illustrates the relationship between quality improvement, reorder point, and lead time, as affected by backorder rate, in an imperfect production process. To reduce the total system cost by optimizing the setup cost, lot size, lead time, reorder point, and process quality parameter simultaneously, we first consider that the lead time demand follows a normal distribution, then we apply the distribution free approach for the lead time demand. We prove two lemmas which are used to find optimal solutions for the basic and distribution free models. We compare our models with the existing models using numerical examples and show that significant savings over the existing models can be achieved.

1. Introduction

Since the invention of the economic order quantity by Harris (1913), many continuous applications of it have been seen in different sectors of everyday life. In Harris's model, most of the values are taken as constants. Silver (1992) suggested '*changing the givens*' concept such that constant terms are regarded as decision variables, and many researchers have used it to obtain new models that incorporate improvement efforts. As a practical matter, not all factors, such as lot size, backorder rate, lead time, and quality control parameter remain constant. Therefore, an enormous amount of research came from different sectors. The researchers faced problems in discussing their models because of the difficult and time-consuming process in determining the lead time demand distribution. Scarf (1958) found a min–max solution of the newsvendor problem in which only the mean and the standard deviation of the lead time demand distribution are assumed to be known. Though beautifully explained, the model is difficult to understand and efficiently implement.

Gallego and Moon (1993) made Scarf's (1958) ordering rule very easy. Moon and Gallego (1994) discussed different ways of applying the distribution free procedure for some inventory models. Ben-Daya and Raouf (1994) developed an inventory model in which the lead time is a decision variable. Moon and Choi (1995) extended the distribution free newsvendor problem to allow customer balking. Ouyang et al. (1996) extended the model of Ben-Daya and Raouf (1994) by adding the lead time demand cost. They considered the total amount of the lead time demand as a mixture of backorders and lost sales during the stockout period. Ouyang and Wu (1997) discussed an inventory model with a service level constraint in which lead time is variable and applied the distribution free approach.

By correcting the model of Ouyang et al. (1996) and Moon and Choi (1998) developed a complete solution algorithm for the model in which the lead time is a decision variable. Ouyang and Wu (1998) discovered both the optimal order quantity and the lead time by taking them as decision variables in a distribution free procedure, thus extending the concept of Ben-Daya and Raouf (1994) and Ouyang et al. (1996). Hariga and Ben-Daya (1999) discussed stochastic inventory models with a deterministic lead time and optimal ordering decision. Ouyang et al. (2002) extended the model of Moon and Choi (1998) by considering quality improvement and setup cost reduction. To reduce total system expense, Chuang et al. (2004) investigated the periodic review inventory model with a mixture of backorders and lost sales by simultaneously controlling the lead time and the setup cost.

Alfares and Elmorra (2005) discussed the extension of the distribution free newsvendor problem with shortages. This model determines an optimal order quantity and a lower bound on the profit under the worst possible distribution of the lead time demand in single product, fixed ordering cost, random yield, and resource-constrained multi-product cases. Chu et al. (2005) corrected the solution algorithm of Ouyang and Wu (1997) by developing lemmas to reveal the parameter's effects and presenting two complete procedures to determine the optimal solution. Wu et al. (2007) discussed a computational algorithmic procedure for the optimal inventory policy involving a negative exponential

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http://dx.doi.org/10.1016/j.ijpe.2013.11.014

crashing cost and lead time demand. Lee and Hsu (2011) derived the effect of advertising costs on the distribution free newsvendor problem. Liao et al. (2011) developed a distribution free newsvendor model with balking and lost sales penalty. Kono (2012) discussed a safe zone analysis for multiple investment alternatives on the total and unit cost domains. Nagasawa et al. (2012) developed the selection of ordering policy and items classification based on the canonical correlation and cluster analysis. Glock (2012) discussed a lead time reduction strategy in an integrated inventory model with lot size-dependent lead times and stochastic demand. Yoo et al. (2012) developed an optimal lot sizing model for an imperfect production and inspection system with customer return and defective disposal. Pal et al. (2013) studied a distribution free newsvendor problem in which the holding cost function depends on order quantity and the inventory level. Cobb (2013) explained a mixture distribution procedure for the lead time demand in a continuous review inventory model.

Most inventory models considered either constant backorder rate or price-dependent backorder rate. However, it is much more realistic to consider backorder rate as either planned or variable. In this direction, Ouyang and Chuang (2001) extended Ouyang et al.'s (1996) model to the model with variable backorder rate. See Table 1 for a comparison of our study with others.

Pan et al. (2004) and Pan and Hsiao (2005) discussed two inventory models based on price-dependent backorder rate. Lee (2005) considered the mixture of distributions model with controllable backorder rate and variable lead time. Lin (2008) developed an inventory model by simultaneously optimizing the order quantity, reorder point, backorder price discount, and lead time. Cárdenas-Barrón (2009) presented an inventory model with rework process at a single-stage production system with planned backorders and obtained a closed-form optimal solution. Chung and Cárdenas-Barrón (2012) developed a complete solution procedure for the economic order quantity and economic production quantity inventory models with linear and fixed backorder costs. Ouyang and Chang (2013) developed a mathematical model to find the optimal production policy for an EPQ inventory system with imperfect production processes under permissible delay in payments and complete backorder rate.

In our proposed model, we extend the study of Ouyang et al. (2002) by adding a variable backorder rate. We use the same assumptions as Ouyang et al. (2002), except for backorder rate, which may not always be constant as they presumed in their study. We consider backorder rate as a function of lead time such that increased shortage values indicate increased loss as customers refuse to wait for delivery. Hence, if the backorder rate is low, the

Table 1
Comparison between the contributions of different authors.

Author (s)	Distribution free approach	Variable lead time	Order quantity	Setup cost reduction	Quality improvement	Variable back order cost	Fixed back order cost	Variable reorder point
Harris (1913)			√					
Scarf (1958)	√		√					
Porteus (1986)			√	√	√			
Gallego and Moon (1993)	√		√					
Moon and Gallego (1994)	√		√				√	√
Ben-Daya and Raouf (1994)		√	√					
Moon and Choi (1995)	√		√					√
Ouyang et al. (1996)		√	√				√	
Ouyang and Wu (1997)	√	√	√				√	
Ouyang and Wu (1998)	√	√	√				√	√
Moon and Choi (1998)	√	√	√				√	√
Hariga and Ben-Daya (1999)	√	√	√				√	√
Ouyang and Chuang (2001)	√	√	√			√		
Ouyang et al. (2002)	√	√	√	√	√		√	√
Chuang et al. (2004)	√	√					√	
Lee et al. (2004)	√	√	√				√	√
Pan et al. (2004)	√	√	√			√[1]		√
Pan and Hsiao (2005)	√	√	√			√[1]		
Chu et al. (2005)	√	√	√				√	
Lee (2005)	√	√	√			√		
Alfares and Elmorra (2005)	√		√				√	
Wu et al. (2007)	√	√	√			√	√	
Lin (2008)	√	√	√				√[1]	√
Cárdenas-Barrón (2009)						√		
Liao et al. (2011)	√		√					
Lee and Hsu (2011)	√		√[2]				√	
Glock (2012)		√	√				√	√
Kono (2012)				√				
Nagasawa et al. (2012)				√				
Chung and Cárdenas-Barrón (2012)						√		
Yoo et al. (2012)			√		√			
Cobb (2013)	√	√						
Ouyang and Chang (2013)			√			√		
Pal et al. (2013)	√		√					
This model	√	√	√	√	√	√		√

√[1] indicates the backorder with price discount offer and √[2] indicates the model with the advertising policy.

company earns more profit. The paper is organized as follows: The mathematical model is formulated in Section 2. In Section 3, the distribution free model is developed. Some numerical examples and sensitivity analysis are given to illustrate the model in Section 4. Finally, we offer conclusions in Section 5.

2. Mathematical model

The following notation is used to develop this model:

Q	order quantity (units) (decision variable)
k	safety factor (decision variable)
A	ordering cost per order (\$/order) (decision variable)
L	replenishment lead time (weeks) (decision variable)
θ	probability of the production process which may go to *out-of-control* state during producing a lot (decision variable)
D	average demand per year (units/year)
A_0	fixed initial ordering cost per order (\$/order)
h	holding cost per unit per year (\$/unit/year)
π	stockout cost per unit short (\$/unit short)
r	reorder point (units)
μ	mean of the lead time demand
σ	standard deviation of the lead time demand
θ_0	initial probability of the production process which may go to *out-of-control* state during producing a lot
α	annual fractional cost of capital investment (\$/year)
$I(A,\theta)$	setup cost reduction and capital investment required to reduce setup cost from A_0 to A and the *out-of-control* probability from θ_0 to θ
L_i	length of the lead time with components $i=1,2,\ldots,n$ (weeks)
u_i	component of the lead time with u_i as the minimum duration (days)
v_i	component of the lead time with v_i as normal duration (days)
c_i	component of the lead time with c_i as crashing cost per unit time (\$/day)
X	lead time demand which has a distribution function F (units)
$E(x)$	mathematical expectation of x
x^+	max $\{x,0\}$
$E(X-r)^+$	expected shortage quantity at the end of the cycle

The following assumptions are considered to develop this model.

1. The lead time L has n mutually independent components, each having a different crashing cost for reducing lead time. The ith component has a normal duration v_i and the minimum duration u_i with crashing cost per unit time c_i with $c_1 \le c_2 \le c_3 \le \cdots \le c_n$. The lead time demand X follows a normal distribution with mean DL and standard deviation $\sigma\sqrt{L}$.
2. Let $L_0=\sum_{j=1}^{n} v_j$ and L_i be the length of the lead time with components $1,2,\ldots,i$ crashed to their minimum duration. Then, L_i can be written as $L_i=L_0-\sum_{j=1}^{i}(v_j-u_j)$ and the lead time crashing cost per cycle $R(L)$ is expressed as $R(L)=c_i(L_i-L)+\sum_{j=1}^{i-1} c_j(v_j-u_j)$ for $i=1,2,3,\ldots,n$.
3. We consider the variable backorder rate β with respect to lead time.
4. Logarithmic expressions are used for both quality improvement and setup cost reduction.
5. The reorder point $r=$ expected demand during the lead time $+$ safety stock (SS) and SS $=k$ (standard deviation of the lead time demand) in which k is a safety factor.

Ouyang et al. (2002) found an optimal solution for setup cost reduction, quality improvement, lot size, reorder point, and lead time for reducing total production costs. The model extended the work of Moon and Choi (1998) in investigating the effects of investment on quality improvement and setup cost reduction. The model assumes the normally distributed lead time demand X with mean DL and standard deviation $\sigma\sqrt{L}$. Based on Moon and Choi (1998), the associated cost of the model is

$$C(Q,r,L)=\frac{AD}{Q}+h\left[\frac{Q}{2}+r-DL+(1-\beta)E(X-r)^+\right]$$
$$+\frac{[\pi+\pi_0(1-\beta)D]}{Q}E(X-r)^+ +\frac{D}{Q}\left\{c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j(v_j-u_j)\right\} \quad (1)$$

After the incorporation of the defective items, the expected annual total cost can be expressed as

$$M(Q,r,L)=C(Q,r,L)+\frac{sDQ\theta}{2} \quad (2)$$

We consider the concept of Porteus (1986) for quality improvement

$$I_\theta(\theta)=b\ \ln\left(\frac{\theta_0}{\theta}\right) \quad \text{for } 0<\theta\le\theta_0 \quad (3)$$

and for setup cost reduction

$$I_A(A)=B\ \ln\left(\frac{A_0}{A}\right) \quad \text{for } 0<A\le A_0 \quad (4)$$

Hence, the total investment for quality improvement and setup cost reduction becomes as follows:

$$I(\theta,A)=I_\theta(\theta)+I_A(A)=G-b\ \ln\theta-B\ \ln A$$
$$\text{where } G=b\ \ln(\theta_0)+B\ \ln(A_0) \quad (5)$$

The aim of the problem is to minimize the cost

$$EAC(Q,r,\theta,A,L)=\alpha(G-b\ \ln\theta-B\ \ln A)+\frac{AD}{Q}$$
$$+h\left[\frac{Q}{2}+r-DL+(1-\beta)E(X-r)^+\right]$$
$$+\frac{[\pi+\pi_0(1-\beta)]D}{Q}E(X-r)^+ +\frac{D}{Q}R(L)+\frac{sDQ\theta}{2}$$
$$\text{for } 0<\theta\le\theta_0 \quad \text{and} \quad 0<A\le A_0 \quad (6)$$

The reorder point is $r=DL+k\sigma\sqrt{L}$ and $E(X-r)^+=\int_r^\infty(x-r)\,dF(x)=\sigma\sqrt{L}\{\phi(k)-k(1-\Phi(k))\}=\sigma\sqrt{L}\psi(k)$, where $\psi(k)=\phi(k)-k(1-\Phi(k))$, $\phi(k)$ and $\Phi(k)$ are the standard normal distribution function and the probability density function of the normal distribution respectively. Therefore, the safety factor k can be treated as a decision variable instead of r. Thus, the problem of Ouyang et al. (2002) becomes as follows in which N represents the normal distribution:

$$\text{Min} \quad EAC^N(Q,k,\theta,A,L)=\alpha(G-b\ \ln\theta-B\ \ln A)+\frac{AD}{Q}+h\left(\frac{Q}{2}+k\sigma\sqrt{L}\right)$$
$$+\left[h(1-\beta)+\frac{D}{Q}(\pi+\pi_0(1-\beta))\right]\sigma\sqrt{L}\psi(k)+\frac{D}{Q}R(L)+\frac{sDQ\theta}{2}$$
$$\text{subject to} \quad 0<\theta\le\theta_0, \quad 0<A\le A_0 \quad (7)$$

In a competitive marketing environment, demand often remains unsatisfied due to long lead times. Hence, a variable backorder rate, based on the lead time to accommodate practical situations, is an important factor. Thus, we consider the backorder rate as a function of the lead time as follows:

$$\beta=\frac{1}{1+\rho\sigma\sqrt{L}\psi(k)}, \quad \rho \text{ being constant}, \quad 0<\rho<\infty \quad (8)$$

Therefore, our problem reduces to

$$\text{Min} \quad EAC^N(Q,k,\theta,A,L) = \alpha(G - b\ln\theta - B\ln A) + \frac{AD}{Q} + h\left(\frac{Q}{2} + k\sigma\sqrt{L}\right)$$
$$+\left[\frac{h\rho\sigma\sqrt{L}\psi(k)}{1+\rho\sigma\sqrt{L}\psi(k)} + \frac{D}{Q}\left(\pi + \frac{\pi_0\rho\sigma\sqrt{L}\psi(k)}{1+\rho\sigma\sqrt{L}\psi(k)}\right)\right]\sigma\sqrt{L}\psi(k) \quad + \frac{D}{Q}R(L) + \frac{sDQ\theta}{2}$$
$$\text{subject to} \quad 0<\theta\le\theta_0, \quad 0<A\le A_0 \tag{9}$$

We minimize the expected associated cost with respect to restrictions. We first ignore the restrictions and solve the non-linear program with an analytical method and calculate all the partial derivatives of the cost function with respect to five decision variables. Then, we apply the restrictions. We obtain the values as follows:

$$\frac{\partial EAC^N(Q,k,\theta,A,L)}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{D\overline{\pi}\xi}{\rho Q^2} - \frac{D}{Q^2}R(L) + \frac{sD\theta}{2} \tag{10}$$

$$\frac{\partial EAC^N(Q,k,\theta,A,L)}{\partial k} = h\sigma\sqrt{L} + \rho\sigma^2 L\left(h + \frac{D\pi_0}{Q}\right)\left[\frac{2\psi(k)\xi_3}{1+\xi} - \frac{(\psi(k))^2\rho\sigma\sqrt{L}\xi_3}{(1+\xi)^2}\right] + \frac{[D\pi\xi_3\sigma\sqrt{L}]}{Q} \tag{11}$$

$$\frac{\partial EAC^N(Q,k,\theta,A,L)}{\partial \theta} = -\frac{\alpha b}{\theta} + \frac{sDQ}{2} \tag{12}$$

$$\frac{\partial EAC^N(Q,k,\theta,A,L)}{\partial A} = -\frac{\alpha B}{A} + \frac{D}{Q} \tag{13}$$

$$\frac{\partial EAC^N(Q,k,\theta,A,L)}{\partial L} = \frac{1}{2}hk\sigma L^{-1/2} + \left(h + \frac{D\pi_0}{Q}\right)\frac{(2+\xi)\xi^2}{2\rho L(1+\xi)^2} + \frac{D\pi\xi}{2\rho LQ} - \frac{D}{Q}c_i \tag{14}$$

where $\overline{\pi} = \pi + \frac{\pi_0\xi}{1+\xi}$, $\xi = \rho\sigma\sqrt{L}\psi(k)$, and $\xi_3 = \Phi(k) - 1$

By investigating the second order sufficient conditions, it is found that $EAC^N(Q,k,\theta,A,L)$ is not a convex function for L because the second order partial derivative of $EAC^N(Q,k,\theta,A,L)$ with respect to L is negative, i.e.,

$$\frac{\partial^2 EAC^N(Q,k,\theta,A,L)}{\partial L^2} = -\left[\frac{1}{4}hk\sigma L^{-3/2} + \frac{(3+\xi)\left(h+\frac{D\pi_0}{Q}\right)\xi^3}{4\rho L^2(1+\xi)^3} + \frac{D\pi\xi}{4\rho QL^2}\right] < 0 \tag{15}$$

If we take the values of Q,k,θ and A as constant, then the function $EAC^N(Q,k,\theta,A,L)$ is concave with respect to L. Therefore, for constant values of Q,k,θ, and A, the minimum expected cost can be obtained from the end points of $[L_i, L_{i-1}]$. Thus, we obtain the optimal values of Q,k,θ, and A for a given $L\in[L_i,L_{i-1}]$. Now equating first four partial derivatives to zero, we obtain

$$Q = \sqrt{\frac{2D\{\rho A + \overline{\pi}\xi + \rho R(L)\}}{\rho(h+sD\theta)}} \tag{16}$$

$$\Phi(k) = 1 + \frac{(1+\xi)^2 hQ}{(D\pi_0 + hQ) - (D\pi + hQ + D\pi_0)(1+\xi)^2} \tag{17}$$

$$\theta = \frac{2\alpha b}{sDQ} \tag{18}$$

$$A = \frac{\alpha BQ}{D} \tag{19}$$

It is a non-linear program. Thus, we construct a lemma to obtain the optimum value.

Lemma 1. *For a given $L\in[L_i,L_{i-1}]$, the Hessian matrix for $EAC^N(Q,k,\theta,A,L)$ is always positive definite at the optimal values (Q^*,k^*,θ^*,A^*).*

Proof. See Appendix A.

From the last equations, it is clear that the values of θ and A are positive. Based on the restrictions on θ and A, we have four conditions for a given $L\in[L_i,L_{i-1}]$ as

1. If $\theta^*<\theta_0$ and $A^*<A_0$, then (Q^*,r^*,θ^*,A^*) is an optimal solution.
2. If $\theta^*\ge\theta_0$ and $A^*<A_0$, then it is not profitable to invest in the quality improvement process, i.e., $\theta^*=\theta_0$.
3. If $\theta^*<\theta_0$ and $A^*\ge A_0$, then the initial setup cost is an optimal setup cost, i.e., $A^*=A_0$.
4. If $\theta^*\ge\theta_0$ and $A^*\ge A_0$, then we do not consider any investment to reduce setup cost or to improve quality, i.e., $\theta^*=\theta_0$ and $A^*=A_0$.

For this non-linear program, we cannot obtain a closed-form solution. Hence, we use the same algorithm developed by Ouyang et al. (2002). After getting the values of L_N and k_N, the optimal reorder point $r_N = DL_N + k_N\sigma\sqrt{L_N}$ can be obtained.

Special Case 1. When ρ is zero, β becomes 1 which is a constant, and this model is reduced to the complete backorder case. Therefore, our result coincides with that of Ouyang et al. (2002). Note that the major contribution of this model is to use a variable backorder rate.

Special Case 2. When ρ is infinity, β becomes 0 and this model is reduced to the no backorder case. This model is exactly the same model with Moon and Choi (1998) with a quality improvement and setup cost reduction policy.

3. Distribution free model

In many practical situations, the lead time distribution is unknown. As a result, we cannot calculate the expected shortages per replenishment cycle $E(X-r)^+$ and instead apply the min–max distribution free approach suggested by Scarf (1958) and made easier by Gallego and Moon (1993). In this manner, we find the least favorable distribution function in $\mathbf{F}$ for each variable (Q,r,θ,A,L) and then minimize the annual total cost with respect to the decision variables. Therefore, the problem is reduced to

$$\text{Min} \quad \underset{F\in\mathbf{F}}{\text{Max}}\, EAC(Q,r,\theta,A,L)$$
$$\text{subject to} \quad 0<\theta\le\theta_0, \quad 0<A\le A_0 \tag{20}$$

Lemma 1 by Gallego and Moon (1993) is applied here with the reorder point as $r = DL + k\sigma\sqrt{L}$. We obtain an upper bound of the expected shortages per replenishment cycle as

$$E(X-r)^+ \le \tfrac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2}-k) \quad \text{for any } F\in\mathbf{F} \tag{21}$$

Moreover, this upper bound is tight by Gallego and Moon (1993). The value of the backorder rate β is expressed by using the above inequality as $\beta \ge \left\{1/(1+\tfrac{1}{2}\rho\sigma\sqrt{L}(\sqrt{1+k^2}-k))\right\}$.

Therefore, our problem reduces to as follows in which w represents the worst distribution.

$$\text{Min} \quad EAC^w(Q,k,\theta,A,L) = \alpha(G - b\ln\theta - B\ln A) + \frac{AD}{Q}$$
$$+h\left(\frac{Q}{2}+k\sigma\sqrt{L}\right) + \frac{\pi D\sigma\sqrt{L}}{2Q}(\sqrt{1+k^2}-k)$$
$$+\frac{1}{2}\left(\frac{\pi_0 D}{Q}+h\right)\left[\frac{\rho\sigma^2 L(\sqrt{1+k^2}-k)^2}{\rho\sigma\sqrt{L}(\sqrt{1+k^2}-k)+2}\right] + \frac{DR(L)}{Q} + \frac{sDQ\theta}{2}$$
$$\text{subject to} \quad 0<\theta\le\theta_0, \ 0<A\le A_0 \tag{22}$$

We first ignore restrictions and find the partial derivatives of $EAC^w(Q,k,\theta,A,L)$. But $EAC^w(Q,k,\theta,A,L)$ is concave with respect to L because $\partial^2 EAC^w(Q,k,\theta,A,L)/\partial L^2<0$, where

$$\frac{\partial^2 EAC^w(Q,k,\theta,A,L)}{\partial L^2}=-\frac{\sigma}{4L\sqrt{L}}\left[\left(\frac{hk+\pi D}{Q}\right)+\left(\frac{\pi_0 D}{Q}+h\right)\left(\frac{\rho^2\sigma^2\xi_*^3(6+\rho\xi_*\sigma\sqrt{L})}{2(2+\rho\sigma\sqrt{L}\xi_*)^3}\right)\right]$$

To find the minimum value, we find the four partial derivatives for a given $L\in[L_i,L_{i-1}]$ as

$$\frac{\partial EAC^w(Q,k,\theta,A,L)}{\partial Q}=\frac{h}{2}+\frac{sD\theta}{2}-\frac{D}{2Q^2}\left[2A+\pi\xi_1+2R(L)+\frac{\pi_0\rho\xi_1^2}{2+\rho\xi_1}\right] \quad (23)$$

$$\frac{\partial EAC^w(Q,k,\theta,A,L)}{\partial k}=h\sigma\sqrt{L}-\frac{\xi_1}{2\sqrt{1+k^2}}\left[\frac{D\pi}{Q}+\left(h+\frac{D\pi_0}{Q}\right)\left\{1-\frac{4}{(2+\rho\xi_1)^2}\right\}\right] \quad (24)$$

$$\frac{\partial EAC^w(Q,k,\theta,A,L)}{\partial \theta}=-\frac{\alpha b}{\theta}+\frac{sDQ}{2} \quad (25)$$

$$\frac{\partial EAC^w(Q,k,\theta,A,L)}{\partial A}=-\frac{\alpha B}{A}+\frac{D}{Q} \quad (26)$$

where $\xi_1=\xi_*\sigma\sqrt{L}$ and $\xi_*=(\sqrt{1+k^2}-k)$.

Equating Eqs. (23)–(26) to zero, we obtain the following:

$$Q=\sqrt{\frac{(2AD+\pi D\xi_1+2DR(L))(2+\rho\xi_1)+\pi_0 D\rho\xi_1^2}{(h+sD\theta)(2+\rho\xi_1)}} \quad (27)$$

$$(\rho\xi_1+2)^2=\frac{(2+\rho\xi_1)^2}{h+\frac{D\pi_0}{Q}}\left[2h+\frac{2h\sigma\sqrt{L}k}{\xi_1}-\frac{D\pi}{Q}\right]+4 \quad (28)$$

$$\theta=\frac{2\alpha b}{sDQ} \quad (29)$$

$$A=\frac{\alpha BQ}{D} \quad (30)$$

Similar to the first case, we check the sufficient conditions using the Hessian matrix for a given L. With considerations of the necessary and sufficient conditions for an optimum value, we obtain the optimal solutions $(Q_w^*,k_w^*,\theta_w^*,A_w^*)$ for which the cost function $EAC^w(Q,k,\theta,A,L)$ has a minimum value.

Lemma 2. *For a given $L\in[L_i,L_{i-1}]$, the Hessian matrix for $EAC^w(Q,k,\theta,A,L)$ is always positive definite at the optimal values (Q^*,k^*,θ^*,A^*).*

Table 2
Lead time data.

Lead time component i	Normal duration v_i (days)	Minimum duration u_i (days)	Unit crashing cost c_i ($/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Proof. See Appendix B.

4. Numerical experiments

Example 1. We use the same data as in the Ouyang et al. (2002) model. The following parametric values are considered in appropriate units: $D=600$ units/year; $\rho=.1$; $A_0=\$200$/setup; $h=\$20$/unit/year; $\pi=\$50$/unit short; $\pi_0=\$150$/unit; $\sigma=7$ units/week; $\mu=11$ units/week; $\theta_0=0.0002$; $s=\$75$/defective unit; $\alpha=0.1$/dollar/year; $b=400$; and $B=5800$. The lead time has three components, which are shown in Table 2.

We denote $EAC_i^N(Q_N,r_N,\theta_N,A_N,L_N)$, $i=1,2$ by $EAC_i^N(\cdot)$. The optimal solutions for the normal distribution are given in Table 3. Because we used the variable backorder rate, we obtained a better result than Ouyang et al. did in 2002 when the minimum cost was determined for L at 4 weeks. For $\beta=1$, the fully backorder case, both models cannot be compared because Ouyang et al. (2002) used a constant backorder rate while this model assumed a variable backorder rate.

Example 2. In this case, the probability distribution is unknown. We have only information about the mean and the standard deviation of lead time demand. We use the same data as in the Ouyang et al. (2002) model. We denote $EAC_i^W(\cdot)$ to be $EAC_i^W(Q_w,r_w,\theta_w,A_w,L_w)$, $i=1,2$. The optimal solutions for the distribution free model are given in Table 4. In the case of the distribution free model, we obtained a better result than Ouyang et al. (2002) did because we used the variable backorder rate.

Tables 3 and 4 show that investment in both quality improvement and setup cost reduction results in a much lower total cost than investment in quality improvement. Thus, our study established '*changing the givens*' as described by Silver (1992). We compare the cost between the first and the second models with the concept of EVAI (Expected Value of Additional Information) that was introduced by Moon and Gallego (1994). We compare the performance of $(Q_w,r_w,\theta_w,A_w,L_w)$ with $(Q_N,r_N,\theta_N,A_N,L_N)$. The corresponding results are $(Q_w,r_w,\theta_w,A_w,L_w)=(118.87,76.86,0.00001496,114.91,4)$ and $(Q_N,r_N,\theta_N,A_N,L_N)=(81.31,69.63,\ 0.0000218,\ 78.60,4)$ for $L=4$. The cost of using $(Q_w,r_w,\theta_w,A_w,L_w)=(118.87,76.86,0.00001496,114.91,4)$ rather than the optimal solution $(Q_N,r_N,\theta_N,A_N,L_N)=(81.31,69.63,0.0000218,78.60,4)$ for the normal distribution is $EAC^N(Q_w,r_w,\theta_w,A_w,L_w)-EAC^N(Q_N,r_N,\theta_N,A_N,L_N)=EAC^N(118.87,76.86,0.00001496,114.91,4)-EAC^N(81.31,69.63,0.0000218,78.60,4)=\$2,833.95-\$2,806.08=\$27.87$.

Here $EAC^N(Q_w,r_w,\theta_w,A_w,L_w)$ is the annual expected cost of using $(Q_w,r_w,\theta_w,A_w,L_w)$ when the actual lead time demand follows a normal distribution. This is the largest amount that we would be willing to pay for the information of the lead time demand distribution. We can reconfirm the robustness of the distribution approach which has been widely proven in many studies (e.g. Moon and Choi, 1995).

Sensitivity analysis: We performed the sensitivity analysis by replacing each key parameter by −50%, −25%, +25%, and +50% one at a time and keeping remaining parameters unchanged. The

Table 3
The optimal solutions for the normal distribution model (L_N in weeks).

Quality improvement and setup cost reduction model				Quality improvement model (fixed setup cost)			
$(Q_N,r_N,\theta_N,A_N,L_N)$	$EAC_1^N(\cdot)$	β	SL	(Q_N,r_N,θ_N,L_N)	$EAC_2^N(\cdot)$	β	SL
(98.52,53.87,0.0000180,95.24,3)	2962.25	0.9718	0.9970	(127.18,52.54,0.00001397,3)	3088.84	0.9640	0.9971
(81.31,69.63,0.0000218,78.60,4)	2806.08*	0.9736	0.9967	(119.10,67.39,0.00001492,4)	2990.97*	0.9621	0.9967
(71.59,99.06,0.0000248,69.20,6)	2807.62	0.9723	0.9960	(115.80,95.72,0.00001535,6)	3028.78	0.9566	0.9961
(68.70,127.24,0.0000258,66.41,8)	2874.08	0.9700	0.9955	(115.24,123.16,0.00001542,8)	3104.90	0.9517	0.9956

Note: SL denotes a service level, which is measured by $1-P(x>r)$.

* indicates the optimal solution.

Table 4
The optimal solutions for the distribution free model (L_w in weeks).

Quality improvement and setup cost reduction model				Quality improvement model (fixed setup cost)			
$(Q_w, r_w, \theta_w, A_w, L_w)$	$EAC_1^w(\cdot)$	β	SL	$(Q_w, r_w, \theta_w, L_w)$	$EAC_2^w(\cdot)$	β	SL
(126.13,59.84,0.00001409,121.93,3)	3503.49	0.8786	0.9890	(144.78,58.18,0.00001228,3)	3562.22*	0.8720	0.9898
(118.87,76.86,0.00001496,114.91,4)	3500.73*	0.8680	0.9872	(140.58,74.46,0.00001265,4)	3572.74	0.8594	0.9884
(122.30,107.65,0.00001454,118.23,6)	3709.82	0.8456	0.9851	(143.21,104.85,0.00001241,6)	3774.34	0.8364	0.9836
(128.73,136.71,0.00001381,124.44,8)	3932.24	0.8260	0.9836	(147.62,133.92,0.00001204,8)	3985.03	0.8173	0.9848

Note: SL denotes a service level, which is measured by $1-P(x>r)$.

* indicates the optimal solution.

Table 5
Sensitivity analysis for the key parameters of Example 1.

Parameters	Changes (in %)	EAC^N
A_0	−50	−14.33
	−25	−5.95
	+25	+4.61
	+50	+8.38
h	−50	−26.96
	−25	−12.24
	+25	+10.83
	+50	+20.73
π	−50	−6.55
	−25	−19.25
	+25	+0.80
	+50	+1.47
π_0	−50	−0.15
	−25	−0.07
	+25	+0.07
	+50	+0.13
s	−50	−0.99
	−25	−0.41
	+25	+0.32
	+50	+0.58
α	−50	−16.15
	−25	−6.78
	+25	+4.73
	+50	+15.88
b	−50	−1.80
	−25	−0.84
	+25	+ 0.75
	+50	+ 1.42
B	−50	−14.47
	−25	−5.93
	+25	+3.85
	+50	+15.10

Table 6
Sensitivity analysis for the key parameters of Example 2.

Parameters	Changes (in %)	EAC^w
A_0	−50	−11.48
	−25	−4.77
	+25	+3.70
	+50	+6.72
h	−50	−29.96
	−25	−13.75
	+25	+12.26
	+50	+23.50
π	−50	−6.00
	−25	−2.86
	+25	+2.63
	+50	+5.08
π_0	−50	−2.40
	−25	−1.14
	+25	+1.04
	+50	+2.00
s	−50	−0.79
	−25	−0.33
	+25	+0.26
	+50	+0.46
α	−50	−11.82
	−25	−4.87
	+25	+3.25
	+50	+5.15
b	−50	−1.66
	−25	−0.78
	+25	+0.71
	+50	+1.35
B	−50	−7.94
	−25	−3.06
	+25	+1.62
	+50	+2.05

effect of changes of the system parameters on the optimal total cost is summarized in Tables 5 and 6 for Examples 1 and 2, respectively.

The percentage change in the optimal cost indicates that some parameters such as A_0, h, π, α, and B are more sensitive than those such as π_0, s, and b to the optimal cost. From the sensitivity analysis results, we know that increasing the value of all key parameters results in the increased optimal cost of the whole system.

For the distribution free case, we obtained the similar results as in the basic case. However, the percentage changes in EAC are somewhat different between two models.

5. Conclusions

We extended Ouyang et al. 's (2002) model by using a variable backorder rate that depends on lead time. We considered the logarithmic investment functions for quality improvement and setup cost reduction with two restrictions. We constructed one lemma with detailed proof for each model to show the global optimum value of the decision variables. We obtained the minimum cost at the optimal values of the decision variables. In the numerical examples, we compared our result with that of Ouyang et al. 's (2002) model and found that our model reduced the total expected cost of the system. This study contributes to the dissemination of "changing the givens" concept which has been widely applied to many production and inventory models to show that continuous improvement efforts are very essential in practice. It can be extended to studies of fuzzy demand, deterioration of products, and so forth. For future research, one might consider investigating delay-in-payments with this model.

Acknowledgements

The authors are grateful to Professor Cárdenas-Barrón and two anonymous referees for their useful comments on earlier versions of the paper. This work was supported by Research Settlement Fund for the new faculty of SNU (Seoul National University).

Appendix A

Proof of Lemma 1. We compute the Hessian matrix at the optimal values for a given L $[L_i, L_{i-1}]$ as follows:

$$H_{ii}=\begin{pmatrix} \frac{\partial^2 EAC^N(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial \theta^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial A^*} \\ \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial \theta^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial A^*} \\ \frac{\partial^2 EAC^N(\cdot)}{\partial \theta^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial \theta^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial \theta^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial \theta^*\partial k^*} \\ \frac{\partial^2 EAC^N(\cdot)}{\partial A^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial A^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial A^*\partial \theta^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial A^{*2}} \end{pmatrix}$$

where $EAC^N(\cdot)=EAC^N(Q^*,k^*,\theta^*,A^*,L)$. The second order partial derivatives at the optimal values are

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^{*2}}=\frac{2D}{Q^{*3}}\left[A^*+\frac{\overline{\pi}\xi}{\rho}+R(L)\right]$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial k^{*2}}=2\rho\sigma^2 L\left(h+\frac{D\pi_0}{Q^*}\right)\left\{\frac{\xi_3^2+(1+\xi)\phi(k^*)\psi(k^*)}{(1+\xi)^3}\right\}+\frac{D\pi\xi}{\rho Q^*}$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial \theta^{*2}}=\frac{\alpha b}{\theta^{*2}}$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial A^{*2}}=\frac{\alpha B}{A^{*2}}$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^*\partial k^*}=\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial k^*\partial Q^*}=\frac{D\sigma\sqrt{L}(1-\Phi(k^*))}{Q^{*2}}\left[\overline{\pi}+\frac{\pi_0\xi}{(1+\xi)^2}\right]$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^*\partial \theta^*}=\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial \theta^*\partial Q^*}=\frac{sD}{2}$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^*\partial A^*}=\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial A^*\partial Q^*}=-\frac{D}{Q^{*2}}$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial k^*\partial \theta^*}=\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial \theta^*\partial k^*}=0$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial k^*\partial A^*}=\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial A^*\partial k^*}=0$$

$$\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial \theta^*\partial A^*}=\frac{\partial^2 EAC^N(Q^*,k^*,\theta^*,A^*,L)}{\partial A^*\partial \theta^*}=0$$

The principal minors at the optimal values are

$$\det(H_{11})=\det\left(\frac{\partial^2 EAC^N(\cdot)}{\partial Q^{*2}}\right)=\frac{2D}{Q^{*3}}\left[A^*+\frac{\overline{\pi}\xi}{\rho}+R(L)\right]>0$$

This first principal minor is greater than zero because all terms are positive:

$$\det(H_{22})=\det\begin{pmatrix} \frac{\partial^2 EAC^N(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial k^*} \\ \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^{*2}} \end{pmatrix}=\omega\tau-\nu^2$$

where

$$\omega=\frac{2D}{Q^{*2}}\left[A^*+\frac{\overline{\pi}\xi}{\rho}+R(L)\right]$$

$$\tau=\left[\frac{2\rho\sigma^2 L}{Q^*}\left(h+\frac{D\pi_0}{Q^*}\right)\left\{\frac{\xi_3^2+(1+\xi)\phi(k^*)\psi(k^*)}{(1+\xi)^3}\right\}+\frac{D\pi\xi}{\rho Q^{*2}}\right]$$

$$\nu=\left[\frac{D\sigma\sqrt{L}\{1-\Phi(k^*)\}}{Q^{*2}}\left\{\overline{\pi}+\frac{\pi_0\xi}{(1+\xi)^2}\right\}\right]$$

After some simplifications, we obtain

$$\overline{\pi}+\frac{\pi_0\xi}{(1+\xi)^2}=\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}$$

and

$$\begin{aligned}\frac{2D\overline{\pi}\xi}{Q^{*2}\rho}&=\frac{D\sigma\sqrt{L}\psi(k^*)}{Q^{*2}}\left(2\pi+\frac{2\pi_0\xi}{1+\xi}\right)\\ &=\frac{D\sigma\sqrt{L}\psi(k^*)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right)+\frac{D\sigma\sqrt{L}\psi(k^*)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi^2}{(1+\xi)^2}\right)\\ &=\frac{D\sigma\sqrt{L}k^*(\Phi(k^*)-1)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right)+\frac{D\sigma\sqrt{L}\phi(k^*)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right)\\ &\quad+\frac{D\sigma\sqrt{L}\psi(k^*)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi^2}{(1+\xi)^2}\right)=k^*\nu+\nu'\end{aligned}$$

Again

$$\begin{aligned}\tau&=2\rho\sigma^2\left(h+\frac{D\pi_0}{Q^*}\right)\left\{\frac{\xi_3^2+(1+\xi)\phi(k^*)\psi(k^*)}{(1+\xi)^3}\right\}+\frac{D\pi\sigma\sqrt{L}\phi(k^*)}{\rho Q^*}\\ &=\frac{1}{1+\xi}\left[\frac{D\pi\sigma\sqrt{L}\phi(k^*)(1+\xi)}{\rho Q^*}+2\rho\sigma^2 L\left(h+\frac{D\pi_0}{Q^*}\right)\left\{\frac{\xi^3+(1+\xi)\phi(k^*)\psi(k^*)}{(1+\xi)^2}\right\}\right]\\ &>\frac{1}{1+\xi}\left[\frac{D\pi\sigma\sqrt{L}\phi(k^*)(1+\xi)}{\rho Q^*}+\frac{2\rho\sigma^2 LD\pi_0}{Q^*}\left(\frac{(1+\xi)\phi(k^*)\psi(k^*)}{(1+\xi)^2}\right)\right]\\ &>\frac{1}{1+\xi}\left[\frac{D\sigma\sqrt{L}\phi(k^*)}{Q^*}\left(\frac{\pi(1+\xi)}{\rho}+\frac{2\pi_0\xi(1+\xi)}{(1+\xi)^2}\right)\right]\\ &>\frac{D\sigma\sqrt{L}\phi(k^*)}{(1+\xi)Q^*}\left[\frac{\pi(1+\xi)}{\rho}+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}+\frac{\pi_0\xi^2}{(1+\xi)^2}\right]\\ &>\frac{D\sigma\sqrt{L}\phi(k^*)}{(1+\xi)Q^*}\left[\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}+\frac{\pi(1+\xi-\rho)}{\rho}+\frac{\pi_0\xi^2}{(1+\xi)^2}\right]\\ &>\frac{D\sigma\sqrt{L}\phi(k^*)}{(1+\xi)Q^*}\left[\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right]+\frac{D\sigma\sqrt{L}\phi(k^*)}{(1+\xi)Q^*}\left[\frac{\pi(1+\xi-\rho)}{\rho}+\frac{\pi_0\xi^2}{(1+\xi)^2}\right]\\ &>\frac{\phi(k^*)}{1+\rho\sigma\sqrt{L}\phi(k^*)+\rho\sigma\sqrt{L}k(\Phi(k^*)-1)}\frac{D\sigma\sqrt{L}}{Q^*}\left[\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right]\\ &\quad+\frac{D\sigma\sqrt{L}\phi(k^*)}{(1+\xi)Q^*}\left[\frac{\pi(1+\xi-\rho)}{\rho}+\frac{\pi_0\xi^2}{(1+\xi)^2}\right]\\ &>\frac{(\Phi(k^*)-1)D\sigma\sqrt{L}}{Q^*}\left[\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right]+\frac{1}{(1+\xi)}\left[\frac{\pi(1+\xi-\rho)}{\rho}+\frac{\pi_0\xi^2}{(1+\xi)^2}\right]>\nu+\nu''\end{aligned}$$

where

$$\nu=\left[\frac{D\sigma\sqrt{L}(\Phi(k^*)-1)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right)\right]$$

$$\nu'=\frac{D\sigma\sqrt{L}\phi(k^*)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right)+\frac{D\sigma\sqrt{L}\psi(k^*)}{Q^{*2}}\left(\pi+\frac{\pi_0\xi^2}{(1+\xi)^2}\right)$$

$$\nu''=\frac{1}{(1+\xi)}\left[\frac{\pi(1+\xi-\rho)}{\rho}+\frac{\pi_0\xi^2}{(1+\xi)^2}\right]>0$$

Thus,

$$\omega=\frac{2D}{Q^{*2}}\left[A^*+R(L)\right]+k^*\nu+\nu'>\nu$$
$$\tau=\nu+\nu''>\nu$$

Hence, $\det(H_{22})>0$:

$$\det(H_{33})=\det\begin{pmatrix}\frac{\partial^2 EAC^N(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial\theta^*}\\ \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial\theta^*}\\ \frac{\partial^2 EAC^N(\cdot)}{\partial\theta^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial\theta^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial\theta^{*2}}\end{pmatrix}$$
$$=\left(\frac{\alpha b}{\theta^{*2}}\omega-\frac{s^2D^2}{4}\right)\tau-\frac{\alpha b}{\theta^{*2}}\nu^2$$

Therefore, it is enough to show

$$\left(\frac{\alpha b}{\theta^{*2}}\omega-\frac{s^2D^2}{4}\right)>\frac{\alpha b}{\theta^{*2}}\nu$$

Now

$$\frac{\alpha b}{\theta^{*2}}\omega-\frac{s^2D^2}{4}=\frac{2D\alpha b}{\theta^{*2}Q^{*2}}\left\{A^*+\frac{\overline{\pi}\xi}{\rho}+R(L)\right\}-\frac{s^2D^2}{4}$$
$$=\frac{\alpha}{\theta^{*2}}\left[\frac{2D}{Q^{*2}}\left\{A^*+\frac{\overline{\pi}\xi}{\rho}+R(L)\right\}-\frac{\alpha b}{Q^{*2}}\right]$$
$$>\frac{\alpha b}{\theta^{*2}}\left[\frac{2D}{Q^{*2}}\left(\frac{\overline{\pi}\xi}{\rho}\right)-\frac{\alpha b}{Q^{*2}}\right]$$
$$>\frac{\alpha b}{\theta^{*2}}\left[\frac{2D\overline{\pi}\xi}{Q^{*2}\rho}-\frac{\alpha b}{Q^{*2}}\right]$$
$$>\frac{\alpha b}{\theta^{*2}}\left[k^*\nu+\nu'-\frac{\alpha b}{Q^{*2}}\right]$$

which can be written as

$$\frac{\alpha b}{\theta^{*2}}\left[k^*\nu+\nu'-\frac{\alpha b}{Q^*}\right]>\frac{\alpha b}{\theta^{*2}}\nu\ \text{if}\ \frac{\alpha b}{\theta^{*2}}\left[\nu'-\frac{\alpha b}{Q^*}\right]>0$$

and

$$\frac{\alpha b}{\theta^{*2}}\left[\nu'-\frac{\alpha b}{Q^*}\right]=\frac{\alpha bD\sigma\sqrt{L}}{Q^{*2}\theta^{*2}}\left[\phi(k^*)\left(\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right)+\psi(k^*)\left(\pi+\frac{\pi_0\xi^2}{(1+\xi)^2}\right)-\frac{\alpha bQ^*}{D\sigma\sqrt{L}}\right]>0$$

This is due to the fact that α is very small with respect to the rest terms in the expression.

Hence, $\det(H_{33})>0$.

Now

$$\det(H_{44})=\det\begin{pmatrix}\frac{\partial^2 EAC^N(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial\theta^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial Q^*\partial A^*}\\ \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial\theta^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial k^*\partial A^*}\\ \frac{\partial^2 EAC^N(\cdot)}{\partial\theta^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial\theta^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial\theta^{*2}} & \frac{\partial^2 EAC^N(\cdot)}{\partial\theta^*\partial k^*}\\ \frac{\partial^2 EAC^N(\cdot)}{\partial A^*\partial Q^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial A^*\partial k^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial A^*\partial\theta^*} & \frac{\partial^2 EAC^N(\cdot)}{\partial A^{*2}}\end{pmatrix}$$
$$=\frac{\alpha B}{A^{*2}}|H_{33}|-\frac{\alpha bD^2\tau}{\theta^{*2}Q^{*4}}$$
$$=\frac{\alpha B}{A^{*2}}\left[\left\{\frac{\alpha b\omega}{\theta^{*2}}-\frac{s^2D^2}{4}-\frac{\alpha^2bB}{\theta^{*2}Q^{*2}}\right\}\tau-\frac{\alpha b\nu^2}{\theta^{*2}}\right]$$
$$=\frac{\alpha^2bB}{A^{*2}\theta^{*2}}\left[\left(\omega-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}\right)\tau-\nu^2\right]$$

It is enough to show

$$\omega-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}>\nu$$

i.e.,

$$\nu'-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}>0$$

Now,

$$\nu'-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}=\frac{D\sigma\sqrt{L}}{Q^{*2}}\left[\phi(k^*)\left(\pi+\frac{\pi_0\xi(2+\xi)}{(1+\xi)^2}\right)+\psi(k^*)\left(\pi+\frac{\pi_0\xi^2}{(1+\xi)^2}\right)-\frac{\alpha(b+B)}{D\sigma\sqrt{L}}\right]>0$$

This is due to the fact that α is very small with respect to the rest terms in the expression. Hence, $\det(H_{44})>0$. Thus, all principal minors are positive which implies that the Hessian matrix H_{ii} is positive definite at (Q^*,k^*,θ^*,A^*). □

Appendix B

Proof of Lemma 2. For a given $L\ \in[L_i,L_{i-1}]$, we consider the Hessian matrix at the optimal values as

$$H_{ii}=\begin{pmatrix}\frac{\partial^2 EAC^w(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial\theta^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial A^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial\theta^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial A^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial k^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial A^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial A^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial A^*\partial\theta^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial A^{*2}}\end{pmatrix}$$

The second order partial derivatives at the optimal values are

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^{*2}}=\frac{D}{Q^{*3}}\left[2A^*+\pi\xi_1+2R(L)+\frac{\pi_0\rho\xi_1^2}{2+\rho\xi_1}\right]$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial k^{*2}}=\frac{D\pi\sigma\sqrt{L}}{2Q^*(1+k^{*2})^{3/2}}+\frac{\left(h+\frac{D\pi_0}{Q^*}\right)\rho\xi_1}{(1+k^{*2})(2+\rho\xi_1)^2}\left[\frac{\sigma\sqrt{L}(4+\rho\xi_1)}{2\sqrt{1+k^{*2}}}+\frac{4\xi_1}{2+\rho\xi_1}\right]$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial\theta^{*2}}=\frac{\alpha b}{\theta^{*2}}$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial A^{*2}}=\frac{\alpha B}{A^{*2}}$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^*\partial k^*}=\frac{D\xi_1}{2Q^{*2}\sqrt{1+k^{*2}}}\left[\pi+\frac{\pi_0\xi_1\rho(4+\rho\xi_1)}{(2+\rho\xi)^2}\right]=\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial k^*\partial Q^*}$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^*\partial\theta^*}=\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial\theta^*\partial Q^*}=\frac{sD}{2}$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial Q^*\partial A^*}=\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial A^*\partial Q^*}=-\frac{D}{Q^{*2}}$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial k^*\partial\theta^*}=\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial\theta^*\partial k^*}=0$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial k^*\partial A^*}=\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial A^*\partial k^*}=0$$

$$\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial\theta^*\partial A^*}=\frac{\partial^2 EAC^w(Q^*,k^*,\theta^*,A^*,L)}{\partial A^*\partial\theta^*}=0$$

At the optimal values, the principal minors are

$$\det(H_{11})=\det\left(\frac{\partial^2 EAC^w(\cdot)}{\partial Q^{*2}}\right)=\frac{D}{Q^{*3}}\left[2A^*+\pi\xi_1+2R(L)+\frac{\pi_0\rho\xi_1^2}{2+\rho\xi_1}\right]>0$$

$$\det(H_{22})=\det\begin{pmatrix}\frac{\partial^2 EAC^w(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial k^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^{*2}}\end{pmatrix}=mn-e^2$$

where

$$m=\det\left(\frac{\partial EAC^w}{\partial Q^{*2}}\right)=\frac{D}{Q^{*3}}\left[2A^*+\pi\xi_1+2R(L)+\frac{\pi_0\rho\xi_1^2}{2+\rho\xi_1}\right]$$

$$n=\det\left(\frac{\partial EAC^w}{\partial k^{*2}}\right)$$
$$=\left[\frac{D\pi\sigma\sqrt{L}}{2Q^*(1+k^{*2})^{3/2}}+\frac{\left(h+\frac{D\pi_0}{Q^*}\right)\rho\xi_1}{(1+k^{*2})(2+\rho\xi_1)^2}\left\{\frac{\sigma\sqrt{L}(4+\rho\xi_1)}{2\sqrt{1+k^{*2}}}+\frac{4\xi_1}{2+\rho\xi_1}\right\}\right]$$

$$e=\det\left(\frac{\partial EAC^w}{\partial Q^*\partial k^*}\right)=\left[\frac{D\xi_1}{2Q^{*2}\sqrt{1+k^{*2}}}\left\{\pi+\frac{\pi_0\xi_1\rho(4+\rho\xi_1)}{(2+\rho\xi)^2}\right\}\right]$$

We have to show $mn-e^2>0$, i.e., if we show $m>e$ and $n>e$, then $H_{22}>0$.

Now we reconstruct m as m_1 and n as n_1 such that $mn=m_1n_1$ as

$$m_1=\frac{D}{Q^{*2}\sqrt{1+k^{*2}}}\left[2\{A^*+R(L)\}+\xi_1\left(\pi+\frac{\pi_0\rho\xi_1(2+\rho\xi_1)}{(2+\rho\xi_1)^2}\right)\right]$$
$$=e+e'$$

where

$$e'=\frac{2D\{A^*+R(L)\}}{Q^{*2}\sqrt{1+k^{*2}}}+\frac{D\xi_1}{2Q^{*2}\sqrt{1+k^{*2}}}\left[\pi+\frac{\pi_0\rho^2\xi_1^2}{(2+\rho\xi_1)^2}\right]$$

and

$$n_1=\frac{D\pi\sigma\sqrt{L}}{2Q^{*2}(1+k^{*2})}+\frac{D\pi_0\rho\xi_1\sigma\sqrt{L}(4+\rho\xi_1)}{2Q^{*2}(1+k^{*2})(2+\rho\xi_1)^2}$$
$$+\frac{h\rho\xi_1}{(1+k^{*2})(2+\rho\xi_1)^2}\left\{\frac{\sigma\sqrt{L}(4+\rho\xi_1)}{2\sqrt{1+k^{*2}}}+\frac{4\xi_1}{(2+\rho\xi_1)}\right\}$$
$$=\frac{D\pi\sigma\sqrt{L}}{2Q^{*2}(1+k^{*2})}+\frac{D\pi_0\rho\xi_1\sigma\sqrt{L}(4+\rho\xi_1)}{2Q^{*2}(1+k^{*2})(2+\rho\xi_1)^2}+e'''$$
$$=\frac{D\pi\sigma\sqrt{L}}{2Q^{*2}(1+k^{*2})}\left[\pi+\frac{D\pi_0\rho\xi_1(4+\rho\xi_1)}{(2+\rho\xi_1)^2}\right]+e'''$$
$$=\frac{\sigma e\sqrt{L}}{\xi_1(\sqrt{1+k^{*2}})}+e'''$$
$$=\frac{(k^*+\sqrt{1+k^{*2}})e}{\sqrt{1+k^{*2}}}+e'''$$
$$=\left(1+\frac{k^*}{\sqrt{1+k^{*2}}}\right)e+e'''$$
$$=e''e+e'''$$

where

$$e'''=\frac{h\rho\xi_1}{(1+k^{*2})(2+\rho\xi_1)^2}\left\{\frac{\sigma\sqrt{L}(4+\rho\xi_1)}{2\sqrt{1+k^{*2}}}+\frac{4\xi_1}{(2+\rho\xi_1)}\right\}$$

$$e''=1+\frac{k^*}{\sqrt{1+k^{*2}}}$$

i.e.,

$$mn=m_1n_1=(e+e')(e''e+e''')$$
$$=e^2+e^{iv}>e^2\quad\text{because } e^{iv}>0.$$

Thus, $\det(H_{22})>0$.

$$\det(H_{33})=\det\begin{pmatrix}\frac{\partial^2 EAC^w(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial\theta^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial\theta^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^{*2}}\end{pmatrix}$$
$$=\frac{\alpha b}{\theta^{*2}}|H_{22}|-n\frac{s^2D^2}{4}$$
$$=\left(\frac{\alpha b}{\theta^{*2}}m-\frac{s^2D^2}{4}\right)n-\frac{\alpha b}{\theta^{*2}}e^2$$
$$=\left(\frac{\alpha b}{\theta^{*2}}e+\frac{\alpha b}{\theta^{*2}}e'-\frac{s^2D^2}{4}\right)(ee''+e''')-\frac{\alpha b}{\theta^{*2}}e^2$$

Therefore, we have to show only the first term is greater than $(\alpha b/\theta^{*2})e$, i.e., it is enough to show that $\alpha be'/\theta^{*2}-s^2D^2/4>0$

Now

$$\frac{\alpha be'}{\theta^{*2}}-\frac{s^2D^2}{4}=\frac{\alpha b}{\theta^{*2}Q^{*2}}\left[\frac{2D(A^*+R(L))}{\sqrt{1+k^{*2}}}+\frac{D\xi_1}{2\sqrt{1+k^{*2}}}\left\{\pi+\frac{\pi_0\rho^2\xi_1^2}{(2+\rho\xi_1)^2}\right\}-\alpha b\right]>0$$

because α is a very small value. Thus, $\det(H_{33})>0$.

$$\det(H_{44})=\det\begin{pmatrix}\frac{\partial^2 EAC^w(\cdot)}{\partial Q^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial\theta^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial Q^*\partial A^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial\theta^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial k^*\partial A^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^{*2}} & \frac{\partial^2 EAC^w(\cdot)}{\partial\theta^*\partial k^*}\\ \frac{\partial^2 EAC^w(\cdot)}{\partial A^*\partial Q^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial A^*\partial k^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial A^*\partial\theta^*} & \frac{\partial^2 EAC^w(\cdot)}{\partial A^{*2}}\end{pmatrix}$$
$$=\frac{\alpha B}{A^{*2}}|H_{33}|-\frac{\alpha bD^2n}{\theta^{*2}Q^{*4}}$$
$$=\frac{\alpha B}{A^{*2}}\left[\left\{\frac{\alpha bm}{\theta^{*2}}-\frac{s^2D^2}{4}-\frac{\alpha^2bB}{\theta^{*2}Q^{*2}}\right\}n-\frac{\alpha be^2}{\theta^{*2}}\right]$$
$$=\frac{\alpha^2bB}{A^{*2}\theta^{*2}}\left[\left(m-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}\right)n-e^2\right]$$

It is enough to show

$$m-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}>e$$

i.e.,

$$e+e'-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}>e$$

i.e.,

$$e'-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}>0$$

Now

$$e'-\frac{\alpha b}{Q^{*2}}-\frac{\alpha B}{Q^{*2}}=\frac{1}{Q^{*2}}\left[\frac{2D(A^*+R(L))}{\sqrt{1+k^{*2}}}+\frac{D\xi_1}{2\sqrt{1+k^{*2}}}\left\{\pi+\frac{\pi_0\rho^2\xi_1^2}{(2+\rho\xi_1)^2}\right\}-\alpha(b+B)\right]>0.$$

because α is a very small value with respect to the rest terms in the expressions. Thus, det $(H_{44})>0$. At the optimal values, all principal minors are positive. Therefore, the Hessian matrix H_{ii} is positive definite at $(Q^*, k^*, \theta^*, A^*)$. □

References

Alfares, H.K., Elmorra, H.H., 2005. The distribution-free newsboy problem: extension to the shortage penalty case. Int. J. Prod. Econ. 93–94, 465–477.
Ben-Daya, M., Raouf, A., 1994. Inventory models involving lead time as decision variable. J. Oper. Res. Soc. 45 (5), 579–582.
Cárdenas-Barrón, L.E., 2009. Economic production quantity with rework process at a single-stage manufacturing system with planned backorders. Comput. Ind. Eng. 57 (3), 1105–1113.
Chu, P., Yang, K.L., Chen, P.S., 2005. Improved inventory models with service level and lead time. Comput. Oper. Res. 32 (2), 285–296.
Chuang, B.R., Ouyang, L.Y., Chung, K.W., 2004. A note on periodic review inventory model with controllable setup cost and lead time. Comput. Oper. Res. 31 (4), 549–561.
Chung, K.J., Cárdenas-Barrón, L.E., 2012. The complete solution procedure for the EOQ and EPQ inventory models with linear and fixed backorder costs. Math. Comput. Model. 55 (11–12), 2151–2156.
Cobb, B.R., 2013. Mixture distributions for modelling demand during lead time. J. Oper. Res. Soc. 64, 217–228.
Gallego, G., Moon, I., 1993. The distribution free newsboy problem: review and extensions. J. Oper. Res. Soc. 44 (8), 825–834.
Glock, C.H., 2012. Lead time reduction strategies in a single-vendor-single-buyer integrated inventory model with lot size-dependent lead times and stochastic demand. Int. J. Prod. Econ. 136 (1), 37–44.
Hariga, M., Ben-Daya, M., 1999. Some stochastic inventory models with deterministic variable lead time. Eur. J. Oper. Res. 113 (1), 42–51.
Harris, F.W., 1913. How many parts to make at once. Fact. Mag. Manag. 10 (2), 135–136, and 152.
Kono, H., 2012. Safer zone analysis for multiple investment alternatives on the total-cost unit-cost domain. Ind. Eng. Manag. Syst. 11 (1), 11–17.
Lee, C.M., Hsu, S.L., 2011. The effect of advertising on the distribution-free newsboy problem. Int. J. Prod. Econ. 129 (1), 217–224.
Lee, W.C., Wu, J.W., Hou, W.B., 2004. A note on inventory model involving variable lead time with defective units for mixtures of distribution. Int. J. Prod. Econ. 89 (1), 31–44.
Lee, W.C., 2005. Inventory model involving controllable backorder rate and variable lead time demand with the mixtures of distribution. Appl. Math. Comput. 160 (3), 701–717.
Liao, Y., Banerjee, A., Yan, C., 2011. A distribution-free newsvendor model with balking and lost sales penalty. Int. J. Prod. Econ. 133 (1), 224–227.
Lin, Y.J., 2008. Minimax distribution-free procedure with backorder price discount. Int. J. Prod. Econ. 111 (1), 118–128.
Moon, I., Choi, S., 1995. The distribution free newsboy problem with balking. J. Oper. Res. Soc. 46 (4), 537–542.
Moon, I., Choi, S., 1998. A note on lead time and distributional assumptions in continuous review inventory models. Comput. Oper. Res. 25 (11), 1007–1012.
Moon, I., Gallego, G., 1994. Distribution free procedures for some inventory models. J. Oper. Res. Soc. 45 (6), 651–658.
Nagasawa, K., Irohara, T., Matoba, Y., Liu, S., 2012. Selecting ordering policy and items classification based on canonical correlation and cluster analysis. Ind. Eng. Manag. Syst. 11 (2), 134–141.
Ouyang, L.Y., Chen, C.K., Chang, H.C., 2002. Quality improvement, setup cost and lead time reductions in lot size reorder point models with an imperfect production process. Comput. Oper. Res. 29 (12), 1701–1717.
Ouyang, L.Y., Wu, K.S., 1997. Mixture inventory model involving variable lead time with a service level constraint. Comput. Oper. Res. 24 (9), 875–882.
Ouyang, L.Y., Wu, K.S., 1998. A minimax distribution-free procedure for mixed inventory model with variable lead time. Int. J. Prod. Econ. 56–57, 511–516.
Ouyang, L.Y., Yeh, N.C., Wu, K.S., 1996. Mixture inventory model with backorders and lost sales for variable lead time. J. Oper. Res. Soc. 47 (6), 829–832.
Ouyang, L.Y., Chuang, B.R., 2001. Mixture inventory model involving variable lead time and controllable backorder rate. Comput. Ind. Eng. 40 (5), 339–348.
Ouyang, L.Y., Chang, C.T., 2013. Optimal production lot with imperfect production process under permissible delay in payments and complete backlogging. Int. J. Prod. Econ. 144 (2), 610–617.
Pal, B., Sana, S.S., Chaudhuri, K.S., 2013. A distribution-free newsvendor problem with nonlinear holding cost. Int. J. Syst. Sci. http://dx.doi.org/10.1080/00207721.2013.815828.
Pan, J.C.H., Lo, M.C., Hsiao, Y.C., 2004. Optimal reorder point inventory models with variable lead time and backorder discount considerations. Eur. J. Oper. Res. 158 (2), 488–505.
Pan, J.C.H., Hsiao, Y.C., 2005. Integrated inventory models with controllable lead time and backorder discount considerations. Int. J. Prod. Econ. 93–94, 387–397.
Porteus, E.L., 1986. Optimal lot sizing, process quality improvement and setup cost reduction. Oper. Res. 34 (1), 137–144.
Scarf, H., 1958. A Min-Max Solution of an Inventory Problem in Studies in the Mathematical Theory of Inventory and Production (201–209). Stanford University Press, Stanford, CA.
Silver, E.A., 1992. Changing the givens in modelling inventory problems: the example of just-in-time systems. Int. J. Prod. Econ. 26 (1–3), 347–351.
Wu, J.W., Lee, W.C., Tsai, H.Y., 2007. Computational algorithmic procedure of optimal inventory policy involving a negative exponential crashing cost and variable lead time demand. Appl. Math. Comput. 184 (2), 798–808.
Yoo, S.H., Kim, D.S., Park, M.S., 2012. Lot sizing and quality investment with quality cost analyses for imperfect production and inspection processes with commercial return. Int. J. Prod. Econ. 140 (2), 922–933.