Inventory systems with variable capacity

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Abstract: Many complex production/inventory systems are characterised by uncertain capacities due to imperfect facilities and processes. Uncertainty in supply capacity may consist of variable supplier capacities and random yields. We extend a model with variable supplier capacity in several directions and analyse the effects of variable supplier capacity. First, we investigate a lot-sizing problem in an EOQ model with variable supplier capacity and random yield. Second, we develop an EOQ model with storage or investment constraints when multiple items are considered. Third, we apply a distribution-free approach (DFA) to the (Q, r) model with variable supplier capacity.
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1 Introduction

Many complex production/inventory systems are characterised by uncertain capacities due to imperfect facilities and processes. In a real life production/inventory system, an assumption that the time of arrival of goods is deterministic or that the received amount exactly equals the quantity ordered may not be tenable. Uncertainty of supply may arise due to variable supplier capacities and random yields. First, we consider that supplier capacity is variable. There are many factors that cause the supplier's capacity to be variable. Unexpected breakdowns and unplanned maintenance may result in down times of uncertain duration; an uncertain duration of repair may affect the availability of the facility, even when the repair is planned; and strikes are possible causes of uncertainty in supply (Ciarallo et al., 1994; Wang and Gerchak, 1996a). Second, we consider that the yield of the item is random. Random yields in a production environment are often due to imperfect processes: a random portion of the items processed turns out to be defective (Yano and Lee, 1995).

Silver (1976) was one of the earliest authors who applied the concept of yield uncertainty to the classical economic order quantity (EOQ) model. He considered an EOQ model where the quantity received is a random proportion of the quantity requisitioned. He obtained the modified EOQ formulas under such a supply uncertainty model. Ciarallo et al. (1994) presented inventory models for a single product with random demand and random capacity. They investigated the problem with single and multiple periods in both finite and infinite horizon settings. For the finite-horizon problem, they showed that the objective function is quasi-convex; thus an optimum exists and in contrast to random-yield models, an 'order-up-to' policy is still optimal. In the single period case, a random capacity does not affect the order-up-to point; this property of the optimal solution is shared with the random-yield model. Wang and Gerchak (1996b) extended the uncertainty model by Ciarallo et al. (1994) by considering a variable capacity and random yield. To minimise the expected total cost (production, holding, and shortage cost), they formulated the problem as a stochastic dynamic programme. Jain and Silver (1995) extended the single period model of Ciarallo et al. (1994) by adding the option of reserving a dedicated capacity level by paying a premium charge to the supplier. They showed that the optimal order size is independent of the random capacity and the optimal dedicated capacity is bounded by zero and the optimal order quantity. Extensive research has been done on lot sizing problems with random yields – see the review article by Yano and Lee (1995). The goal of their paper was to classify and describe the research as of their writing on lot sizing in the presence of random yields. Wang and Gerchak (1996a) analysed the effects of variable capacity on optimal lot sizing in continuous review environments. They considered two models: the basic EOQ model and the order quantity/reorder point model with backlogging. Güllü et al. (1999) considered a periodic review, single-item inventory model under supply uncertainty. They modelled the uncertainty in supply using a three-point mass function, such as completely available, partially available, or unavailable. Hariga and Haouari (1999) derived some simple data-dependent bounds for both the optimal lot size and the expected cost per unit time. They used the uniform, exponential, and truncated normal distributions for the random capacity. Note that when the random entity is yielded, the order quantity/reorder point scenario has been analysed by Noori and Keller (1986) and Gerchak (1992). In his survey article, Silver (2008) classified inventory management problems into eight dimensions, and the nature of supply process including random yields occupied one dimension.

In practice, distributional information on demand is very limited. Sometimes all that is available is an educated guess of the mean μ and the variance σ^2. There is a tendency to use the normal distribution under these conditions. However, the normal distribution does not provide the best protection against the occurrence of other distributions with the same mean μ and same variance σ^2. Scarf (1958) addressed the newsboy problem where only the mean μ and the variance σ^2 of the demand are known without any further assumptions about the form of the distribution of the demand. Taking a conservative approach, he modelled the problem as that of finding the order quantity that maximises the expected profit against the worst possible distribution of the demand with the specified mean and variance. This approach is called a 'distribution-free approach (DFA)'. Gallego (1992) applied DFA to a (Q, r) model with backorder costs. He derived a closed-form solution by which DFA can be applied easily to other inventory models. Gallego and Moon (1993) presented a very compact proof of the optimality of Scarf's (1958) rules for ordering in the newsboy problem. Moon and Choi (1994) solved the distribution free continuous review inventory model with a service level constraint and developed an iterative procedure to find the optimal quantity and reorder point. Moon and Gallego (1994) applied DFA to several important models such as the continuous review (Q, r) model and the periodic review (R, T) model. Moon and Choi (1995) maximised the expected profit against the worst possible distribution of demand for DFA under balking. Moon and Choi (1997) applied DFA to the make-to-order (MTO), make-in-advance (MIA) and composite policies in a single period two echelon stochastic model. Ramaekers and Janssens (2008) developed a procedure to determine shape characteristics when only the first two moments of the distribution of demand during the lead time are known. Moon and Choi (1998) applied DFA to a continuous review inventory model, where they allowed the lead time to be reduced by cost function for lead time crashing. Silver and Moon (2001) solved a recycling problem in which there is available stock that can be converted into end items by utilising the DFA.

We extend a model with variable supplier capacity in several directions and analyse the effects of variable supplier capacity. We formulate an EOQ model with uncertain supply in Section 2. There are two main types of uncertainty: variable supplier capacity and random yield. We extend the uncertainty of Wang and Gerchak (1996a) by decomposing supply uncertainty into variable supplier capacity and random yield. We develop a multi-item EOQ model with a storage or investment constraint in Section 3. In Section 4, we apply the DFA to a (Q, r) model with variable supplier capacity. In Section 5, numerical examples are provided and sensitivity analysis with respect to the parameters is carried out. Finally, conclusions are presented in Section 6.

2 EOQ model with variable supplier capacity and random yield

2.1 EOQ model with variable supplier capacity

In this section, a lot sizing model that considers a general probability function and supply capacity is formulated. We assume that the available capacity for each production/replenishment attempt, which is denoted by u, is a random variable with a probability density function, $g(\cdot)$, and a cumulative distribution, $G(\cdot)$. This random capacity is assumed to be independent of the order quantity/lot size. Thus, when a lot of size Q is released, the realised output or delivered replenishment quantity, Y, will be given by:

$$Y = \ min\ \{Q, u\} = \begin{cases} Q & \text{if} \quad Q \leq u, \\ u & \text{if} \quad Q > u \end{cases}$$

That is, in the presence of uncertainty in the procurement process due to a random available capacity, it is clear that the actual quantity received is the minimum of the quantity ordered and the available capacity. Therefore, the expected actual quantity is:

$$E[Y] = Q \int_{Q}^{\infty} g(u)\mathrm{d}u + \int_{0}^{Q} ug(u)\mathrm{d}u$$

We assume that the proportional cost is applied only to the actually executed production or delivered quantity, i.e., the total proportional production/replenishment costs per cycle is $c \cdot Y$. We will use the above assumption in each model. Note that this assumption is quite relevant. For example, suppliers to a major motor company provide their products at the proportional cost. Meanwhile, some suppliers should pay the penalty if they cannot satisfy the order quantity of the major motor company. In this case, it is the same effect that a discounted cost is applied if the actually executed production or delivered quantity is less than the order quantity. Additional assumptions will be introduced in the paper whenever needed.

We use the following basic notation in subsequent modelling:

A	ordering (set up) cost
c	unit procurement (production) cost
h	holding cost per unit time per item
D	demand rate per unit time
T	length of a cycle
$E[T(\cdot)]$	expected duration of a cycle
C	total cost per cycle
$E[C(\cdot)]$	expected total cost per cycle
$V(\cdot)$	expected cost per unit time, i.e., $V(Q) = \frac{E[C(Q)]}{E[T(Q)]}$

Q	order quantity
u	available supplier capacity, a random variable
$g(\cdot)$	probability density function of the available capacity
$G(\cdot)$	cumulative distribution function of the available capacity.

Additional notation will be introduced in the paper whenever needed.

In the basic EOQ model, it is assumed that demand is constant and deterministic, quantity discounts are not allowed, the replenishment lead time is zero, and the available supplier capacity is a continuous random variable. Under variable capacity, the expected cost per cycle is:

$$\begin{aligned} E[C(Q)] = &\left(A + cQ + \frac{h}{2D}Q^2\right)\int_Q^\infty g(u)\mathrm{d}u \\ &+ \int_0^Q \left(A + cu + \frac{h}{2D}u^2\right) g(u)\mathrm{d}u \end{aligned} \tag{1}$$

The expected duration of the cycle is:

$$E[T(Q)] = \frac{1}{D}Q\int_Q^\infty g(u)\mathrm{d}u + \frac{1}{D}\int_0^Q ug(u)\mathrm{d}u \tag{2}$$

Using a result of renewal processes, Kao (1997) proved that the expected long-run average cost is the expected cost that is incurred during a cycle divided by the expected duration of a cycle. Therefore, we have:

$$V(Q) = \frac{E[C(Q)]}{E[T(Q)]}$$

Thus,

$$V(Q) = cD + \frac{2AD + h\left(Q^2\int_Q^\infty g(u)\mathrm{d}u + \int_0^Q u^2 g(u)\mathrm{d}u\right)}{2\left(Q\int_Q^\infty g(u)\mathrm{d}u + \int_0^Q ug(u)\mathrm{d}u\right)}$$

Wang and Gerchak (1996a), besides Hariga and Haouari (1999), proved that the objective function $V(Q)$ is quasi-convex in Q and a unimodal function. The optimal lot size is the solution to the first order condition that is obtained by setting the derivative of the expected cost per unit time to zero, i.e.,

$$Q^2\int_Q^\infty g(u)\mathrm{d}u + 2Q\int_0^Q ug(u)\mathrm{d}u - \int_0^Q u^2 g(u)\mathrm{d}u - \frac{2AD}{h} = 0$$

Due to the complexity of $V(Q)$, a closed-form solution for the optimal order quantity can be hardly obtained. The solution often involves a non-linear equation and/or numerical integration.

If the variable capacity is exponentially distributed, the optimal solution can be easily obtained. That is, if the variable capacity is exponentially distributed with a mean of β, i.e., $g(u) = (1/\beta)exp(-u/\beta)$, $G(u) = 1 - exp(-u/\beta)$ for $u \geq 0$, then the optimal order quantity satisfies the following equation:

$$2\beta^2[exp(-Q/\beta) - 1] + 2\beta Q - \frac{2AD}{h} = 0$$

2.2 *Extension to random yields*

Another element of randomness is introduced into the model by assuming that each delivered lot may contain a random number of defective units. Let R be a random variable that represents the yield rate. R is assumed to be independent of Q and has a probability density function, $f(\cdot)$, and a cumulative distribution, $F(\cdot)$. The actual output of usable items will be given by $R \cdot min\{u, Q\}$. The expected cost per cycle is

$$E[C_R(Q)] = A + c\int_0^\infty \int_Q^\infty Qrg(u)f(r)\mathrm{d}u\mathrm{d}r + c\int_0^\infty \int_0^Q urg(u)f(r)\mathrm{d}u\mathrm{d}r$$
$$+\frac{h}{2D}\left\{\int_0^\infty \int_Q^\infty (Qr)^2 g(u)f(r)\mathrm{d}u\mathrm{d}r + \int_0^\infty \int_0^Q (ur)^2 g(u)f(r)\mathrm{d}u\mathrm{d}r\right\}$$

The expected duration of the cycle is:

$$E[T_R(Q)] = \frac{1}{D}\int_0^\infty \int_Q^\infty Qrg(u)f(r)\mathrm{d}u\mathrm{d}r + \frac{1}{D}\int_0^\infty \int_0^Q urg(u)f(r)\mathrm{d}u\mathrm{d}r$$

Thus, the expected cost per unit time is given by:

$$V_R(Q) = \frac{E[C_R(Q)]}{E[T_R(Q)]}$$

Substituting $E[T_R(Q)]$ and $E[C_R(Q)]$,

$$V_R(Q) = cD$$
$$+\frac{AD + \frac{h}{2}\left\{\int_0^\infty \int_Q^\infty (Qr)^2 g(u)f(r)\mathrm{d}u\mathrm{d}r + \int_0^\infty \int_0^Q (ur)^2 g(u)f(r)\mathrm{d}u\mathrm{d}r\right\}}{\int_0^\infty \int_Q^\infty Qrg(u)f(r)\mathrm{d}u\mathrm{d}r + \int_0^\infty \int_0^Q urg(u)f(r)\mathrm{d}u\mathrm{d}r}$$

To minimise the objective function $V_R(Q)$, we differentiate $V_R(Q)$ with respect to Q.

$$\frac{\mathrm{d}V_R(Q)}{\mathrm{d}Q} = \frac{(h/2)\int_Q^\infty g(u)\mathrm{d}u \int_0^\infty rf(r)\mathrm{d}r M_R(Q)}{\left\{\int_0^\infty \int_Q^\infty Qrg(u)f(r)\mathrm{d}u\mathrm{d}r + \int_0^\infty \int_0^Q urg(u)f(r)\mathrm{d}u\mathrm{d}r\right\}^2}$$

where $M_R(Q)$ is defined by:

$$M_R(Q) = Q^2\int_0^\infty \int_Q^\infty r^2 g(u)f(r)\mathrm{d}u\mathrm{d}r + 2Q\int_0^\infty \int_0^Q r^2 ug(u)f(r)\mathrm{d}u\mathrm{d}r$$
$$-\int_0^\infty \int_0^Q r^2u^2 g(u)f(r)\mathrm{d}u\mathrm{d}r - \frac{2AD}{h}$$

Due to the complexity of the second derivative of the expected total cost function, $V_R(Q)$, we use $M_R(Q)$ to prove that $V_R(Q)$ is a convex function.

Lemma: The expected total cost function $V_R(Q)$ is quasi-convex in Q and a unimodal function.

Proof: Differentiating $M(Q)$, we get:

$$\frac{\mathrm{d}M_R(Q)}{\mathrm{d}Q} = 2Q\int_0^\infty \int_Q^\infty r^2 g(u)f(r)\mathrm{d}u\mathrm{d}r + 2\int_0^\infty \int_0^Q r^2 u f(r)g(u)\mathrm{d}u\mathrm{d}r \geq 0$$

Thus, $M_R(Q)$ is a monotonically increasing function of Q. Furthermore, we can prove that $M_R(0)= -\frac{2AD}{h} < 0$ and $M_R(\infty) = \infty > 0$ as $Q \to \infty$. Thus, there exists Q^*, which satisfies $M_R(Q^*) = 0$. This implies $M_R(Q) < 0$ over the interval $[0, Q^*)$ and positive elsewhere. Note that $V_R(Q)$ is only effected by $M_R(Q)$, because of $(h/2)$ ≥ 0, $(Q\int_0^\infty \int_Q^\infty rg(u)f(r)\mathrm{d}u\mathrm{d}r + \int_0^\infty \int_0^Q urg(u)f(r)\mathrm{d}u\mathrm{d}r)^2 \geq 0$. That is, these terms do not affect the sign of the first derivative of $V_R(Q)$. Therefore, $V_R(Q)$ is decreasing in $Q < Q^*$ and increasing in $Q > Q^*$.

The optimal order quantity satisfies the following equation,

$$\frac{\mathrm{d}V_R(Q)}{\mathrm{d}Q} = M_R(Q) = 0$$

That is,

$$\begin{aligned} &Q^2\int_0^\infty \int_Q^\infty r^2 g(u)f(r)\mathrm{d}u\mathrm{d}r + 2Q\int_0^\infty \int_0^Q r^2 u f(r)g(u)\mathrm{d}u\mathrm{d}r \\ &\quad - \int_0^\infty \int_0^Q r^2 u^2 f(r)g(u)\mathrm{d}u\mathrm{d}r - \frac{2AD}{h} = 0 \end{aligned} \tag{3}$$

If the yield rate, R, is constant, then:

$$Q^2\int_Q^\infty g(u)\mathrm{d}u + 2Q\int_0^Q ug(u)\mathrm{d}u - \int_0^Q u^2 g(u)\mathrm{d}u - \frac{2AD}{hR^2} = 0$$

The above model is the same as that of Wang and Gerchak (1996a). It is difficult to calculate the expectation of R^2 in equation (3), i.e., $\int_0^\infty r^2 f(r)\mathrm{d}r$. However, when the yield rate is uniformly distributed, we are able to employ a very simple procedure to calculate the optimal order quantity. If the yield rate, R, is uniformly distributed over the interval, $[a, b]$, i.e., $f(r) = 1/(b-a)$, then the expectation of R^2 is:

$$E[R^2] = \int_a^b r^2 f(r)\mathrm{d}r = \frac{1}{3}(a^2 + ab + b^2) \tag{4}$$

We can show that the optimal order quantity is reduced by substituting equation (4) into equation (3). That is:

$$\begin{aligned} &\frac{1}{3}Q^2 exp(-Q/\beta)(a^2+ab+b^2) + \frac{2}{3}Q(a^2+ab+b^2)\{\beta - Qexp(-Q/\beta) \\ &\quad -\beta exp(-Q/\beta)\} - \frac{1}{3}(a^2+ab+b^2)\{2\beta^2 - Q^2 exp(-Q/\beta) \\ &\quad -2Q\beta exp(-Q/\beta) - 2\beta^2 exp(-Q/\beta)\} - \frac{2AD}{h} = 0. \end{aligned} \tag{5}$$

A simple bisection algorithm can be used to find the optimal value of Q.

Bisection algorithm

Step 1 Let the left-hand side of equation (5) be $W(Q)$. Define $Q_{low} = 0$ and $Q_{high} = n\sqrt{\frac{2AD}{h}}$ where n is a large integer.

Step 2 Start with an arbitrary $Q > 0$. For example, $Q = \sqrt{\frac{2AD}{h}}$.

Step 3 Compute $W(Q)$ using equation (5).

Step 4

If $W(Q) > 0$, then $Q_{high} \leftarrow Q$ and $Q = \frac{Q_{high}+Q_{low}}{2}$. Go to Step 3.
If $W(Q) < 0$, then $Q_{low} \leftarrow Q$ and $Q = \frac{Q_{high}+Q_{low}}{2}$. Go to Step 3.
If $W(Q) = 0$, we have found an optimal solution.

3 Multi-item EOQ model

Most inventory systems stock more than one item in which each item may not be treated individually because there are interactions between items. For example, warehouse capacity may be limited, forcing items to compete for floor space; there may be an upper limit on the number of orders that may be placed per year; or there may be an upper limit on the maximum dollar investment that is allowed in inventory (Silver et al., 1998). We will now consider the EOQ model with variable supplier capacity in the presence of a budget constraint on the investment in inventory.

The notation to be used is as follows:

i index set of items $\{1, \cdots, n\}$

n number of items

B constraint on the overall budget.

We want to find the optimal order quantity that minimises the expected cost per unit time in relation to the variable supplier capacity without exceeding the budget limit, B. The objective function can be represented as follows:

$$Min \quad V(Q_1, \cdots, Q_n) = \sum_{i=1}^{n} \left(c_i D_i + \frac{A_i D_i + (h_i/2) Q_i^2 \int_{Q_i}^{\infty} g_i(u_i) \mathrm{d}u_i + (h_i/2) \int_0^{Q_i} u_i^2 g_i(u_i) \mathrm{d}u_i}{Q_i \int_{Q_i}^{\infty} g_i(u_i) \mathrm{d}u_i + \int_0^{Q_i} u_i g_i(u_i) \mathrm{d}u_i} \right)$$

The budget constraint is given by:

$$\sum_{i=1}^{n} c_i \{ Q_i \int_{Q_i}^{\infty} g_i(u_i) \mathrm{d}u_i + \int_0^{Q_i} u_i g_i(u_i) \mathrm{d}u_i \} \leq B$$

We can prove that the above objective function is quasi-convex and attains its global minimum (Hariga and Haouari, 1999; Wang and Gerchak, 1996a). If we can find a Karush-Kuhn-Tucker (KKT) solution, it will be a global minimum because the objective function is convex and to be minimised, and the set of feasible solutions is convex. The Lagrangian function is then,

$$L(Q_1,\cdots,Q_n) = V(Q_1,\cdots,Q_n) \\ +\lambda\left(\sum_{i=1}^{n} c_i\{Q_i\int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i + \int_0^{Q_i} u_i g_i(u_i)\mathrm{d}u_i\} - B\right)$$

Then, the KKT conditions are as follows:

$$\frac{\partial L}{\partial Q_i} = \frac{(h_i/2)\int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i M_i(Q_i)}{\{Q_i\int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i + \int_0^{Q_i} u_i g_i(u_i)\mathrm{d}u_i\}^2} + \lambda c_i[1-G_i(Q_i)] = 0 \qquad \forall i$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} c_i\{Q_i\int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i + \int_0^{Q_i} u_i g_i(u_i)\mathrm{d}u_i\} = B \tag{6}$$

where

$$M_i(Q_i) = Q_i^2\int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i + 2Q_i\int_0^{Q_i} u_i g_i(u_i)\mathrm{d}u_i - \int_0^{Q_i} u_i^2 g_i(u_i)\mathrm{d}u_i \\ -\frac{2A_iD_i}{h_i}$$

Solving the above equations, we obtain λ as:

$$\lambda = -\frac{(h_i/2)M_i(Q_i)}{c_i\{Q_i\int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i + \int_0^{Q_i} u_i g_i(u_i)\mathrm{d}u_i\}^2}$$

Since the Lagrangian multiplier, λ, is non-negative, $M_i(Q_i)$ must be non-positive ($M_i(Q_i) \leq 0$). If we differentiate $M_i(Q_i)$ with respect to Q_i, we get:

$$\frac{\mathrm{d}M_i(Q_i)}{\mathrm{d}Q_i} = 2Q_i\int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i + 2\int_0^{Q_i} u_i g_i(u_i)\mathrm{d}u_i \; > \; 0 \qquad \forall i$$

Therefore, $M_i(Q_i)$ is strictly increasing in Q_i. Hence, there exists a unique and finite positive value of Q_i such that Q_i^*.

$$M_i(Q_i^*) = 0 \qquad \forall i$$

Thus, the feasible solution of Q_i is less than or equal to Q_i^*. If the variable capacity is exponentially distributed with a mean of β_i, i.e., $g_i(u_i) = (1/\beta_i)exp(-u_i/\beta_i)$ for $u_i \geq 0$, then we can show that equation (6) can be reduced to:

$$\frac{h_i}{2}\left\{2\beta_i^2[exp(-Q_i/\beta_i)-1] + 2\beta_i Q_i - \frac{2A_iD_i}{h_i}\right\} \\ +\lambda c_i\beta_i^2\left\{1-exp(-Q_i/\beta_i)\right\}^2 = 0$$

A line search algorithm can be used to find the optimal values of λ and Q_i.

Line search algorithm

Step 1 First find $Q_i(i = 1, 2, \cdots, n)$ from the following equation:

$$2\beta_i^2[exp(-Q_i/\beta_i) - 1] + 2\beta_i Q_i - \frac{2A_iD_i}{h_i} = 0 \qquad \forall i \tag{7}$$

If they satisfy the budget constraint, they are optimal. Otherwise, go to Step 2.

Step 2 Start with an arbitrary $\lambda > 0$.

Step 3 Compute Q_i from the following:

$$\begin{aligned}&\frac{h_i}{2}\left\{2\beta_i^2[exp(-Q_i/\beta_i - 1] + 2\beta_i Q_i - \frac{2A_iD_i}{h_i}\right\}\\ &\quad +\lambda c_i\beta_i^2\left\{1 - exp(-Q_i/\beta_i)\right\}^2 = 0 \qquad \forall i\end{aligned} \tag{8}$$

Step 4

If $\sum_{i=1}^{n} c_i\beta_i\{1 - exp(-Q_i/\beta_i)\} < B$, then decrease λ and go to Step 3.
If $\sum_{i=1}^{n} c_i\beta_i\{1 - exp(-Q_i/\beta_i)\} > B$, then increase λ and go to Step 3.
If $\sum_{i=1}^{n} c_i\beta_i\{1 - exp(-Q_i/\beta_i)\} = B$, we have found an optimal solution.

4 The (Q, r) model with a DFA

The (Q, r) model is a continuous review model. A fixed quantity, Q, is ordered whenever the inventory position drops to (or below) the reorder point, r. This model requires information on the lead time demand (Johnson, 1974). However, information about the form of the probability distribution of the lead time demand is often limited in practice. Sometimes, all that is available is an educated guess of the mean and variance. In this section, we apply the DFA to forecast the lead time demand without assuming any particular distribution. We will use the following additional notation:

r	reorder point
p	shortage penalty cost per unit item
X	random demand during a replenishment lead time
$f(x)$	density function of X
$F(x)$	cumulative distribution of X
μ	mean of X
σ^2	variance of X.

First, we consider the basic (Q, r) model with variable supplier capacity. The realised cost per cycle is:

$$C(Q,r) = A + cY + h\left(\frac{Y}{2} + r - \mu\right)\frac{Y}{D} + p\eta(r)$$
$$= \begin{cases} A + cQ + \frac{hQ}{D}(\frac{Q}{2} + r - \mu) + p\eta(r) & \text{if } Q \leq u. \\ A + cu + \frac{hu}{D}(\frac{u}{2} + r - \mu) + p\eta(r) & \text{if } Q > u. \end{cases}$$

where $\eta(r)$ is the expected shortage per cycle, given by:

$$\eta(r) = \int_r^\infty (x - r) f(x)\mathrm{d}x$$

Thus,

$$E[C(Q,r)] = A + p\eta(r) + cQ\int_Q^\infty g(u)\mathrm{d}u + c\int_0^Q ug(u)\mathrm{d}u$$
$$+ \frac{hQ}{D}\left(\frac{Q}{2} + r - \mu\right)\int_Q^\infty g(u)\mathrm{d}u$$
$$+ \frac{h}{D}\int_0^Q u\left(\frac{u}{2} + r - \mu\right)g(u)\mathrm{d}u$$

Further, under variable capacity, the realised duration of a cycle is given as:

$$T(Q) = \frac{Y}{D} = \begin{cases} Q/D & \text{if } Q \leq u, \\ u/D & \text{if } Q > u \end{cases}$$

Thus, the expected duration of the cycle is:

$$E[T(Q)] = \frac{1}{D}Q\int_Q^\infty g(u)\mathrm{d}u + \frac{1}{D}\int_0^Q ug(u)\mathrm{d}u$$

Using a property of renewal processes (Kao, 1997), the expected cost per unit time is obtained as:

$$V(Q,r) = \frac{E[C(Q,r)]}{E[T(Q)]}$$

Using the above equation, we have:

$$\begin{aligned} V(Q,r) = cD + \Big\{&hQ(Q/2 + r - \mu)\textstyle\int_Q^\infty g(u)\mathrm{d}u \\ &+ h\textstyle\int_0^Q u(u/2 + r - \mu)g(u)\mathrm{d}u + D[A + p\eta(r)]\Big\} \\ &/ \Big\{Q\textstyle\int_Q^\infty g(u)\mathrm{d}u + \int_0^Q ug(u)\mathrm{d}u\Big\} \end{aligned} \tag{9}$$

Now, we consider the DFA. We make no assumption on the cumulative distribution function, F, of X other than saying that it belongs to the class $\mathcal{F}$ of cumulative distribution functions with mean μ and variance σ^2. Since the distribution F of X is unknown, we want to minimise the total expected annual cost against the worst possible distribution in $\mathcal{F}$ (Gallego, 1992; Gallego and Moon, 1993; Moon and Gallego, 1994). We can represent the expected shortage per cycle as follows:

$$\eta(r) = E[X-r]^+ \leq \frac{1}{2}\left\{\sqrt{\sigma^2 + (r-\mu)^2} - (r-\mu)\right\}$$

where we let $x^+ = max\{x, 0\}$.

Then,

$$V(Q,r) = cD + \left\{hQ(Q/2 + r - \mu)\int_Q^\infty g(u)\mathrm{d}u + h\int_0^Q u(Q/2 + r - \mu)g(u)\mathrm{d}u + D[A + pE(X-r)^+]\right\} \Big/ \left\{Q\int_Q^\infty g(u)\mathrm{d}u + \int_0^Q ug(u)\mathrm{d}u\right\}$$

Taking the partial derivatives of $V(Q,r)$ with respect to Q, we obtain:

$$\frac{\partial V(Q,r)}{\partial Q} = \frac{(h/2)\int_Q^\infty g(u)\mathrm{d}uM(Q)}{\left\{Q\int_Q^\infty g(u)\mathrm{d}u + \int_0^Q ug(u)\mathrm{d}u\right\}^2} \tag{10}$$

where

$$M(Q) = Q^2\int_Q^\infty g(u)\mathrm{d}u + 2Q\int_0^Q ug(u)\mathrm{d}u - \int_0^Q u^2 g(u)\mathrm{d}u - \frac{2D}{h}\left[A + pE(X-r)^+\right]$$

Due to the complexity of the second derivative of the expected total cost function, $V(Q,r)$, we use $M(Q)$ to prove that $V(Q,r)$ is a convex function in Q.

Lemma: The expected total cost function, $V(Q,r)$, is quasi-convex in Q for any given r.

Proof: Differentiating $M(Q)$ respect to Q, we get:

$$\frac{dM(Q)}{dQ} = 2Q\int_Q^\infty g(u)\mathrm{d}u + 2\int_0^Q ug(u)du > 0$$

Hence, $M(Q)$ is strictly increasing in Q. Furthermore, $M(0) = -2D[A + pE(X - r)^+/h] < 0$ and $M(Q) \to \infty$ as $Q \to \infty$. Thus, there exists a unique and finite positive value of Q, denoted by Q^*, which satisfies:

$$M(Q^*) = 0$$

such that $M(Q) \geq 0$ if $Q \geq Q^*$ and $M(Q) < 0$ if $Q < Q^*$.

Now, noting that the denominator in equation (10) is always positive and the factor $h\int_Q^\infty g(u)\mathrm{d}u/2$ in its numerator is non-negative, we have:

$$\left[\frac{\partial V(Q,r)}{\partial Q}\right]_{Q<Q^*} \leq 0 \quad and \quad \left[\frac{\partial V(Q,r)}{\partial Q}\right]_{Q\geq Q^*} \geq 0$$

Thus, we have proved that our objective function $V(Q,r)$ in equation (9) is quasi-convex in Q for any given r.

Taking the partial derivative of $V(Q,r)$ with respect to r, we get:

$$\frac{\partial V(Q,r)}{\partial r} = \frac{hQ\int_Q^\infty g(u)\mathrm{d}u + h\int_0^Q ug(u)\mathrm{d}u + \frac{1}{2}Dp\left\{\frac{r-\mu}{\sqrt{\sigma^2+(r-\mu)^2}} - 1\right\}}{Q\int_Q^\infty g(u)\mathrm{d}u + \int_0^Q ug(u)\mathrm{d}u} \tag{11}$$

and

$$\frac{\partial^2 V(Q,r)}{\partial r^2} = \frac{\frac{1}{2}Dp\sigma^2}{\{\sigma^2+(r-\mu)^2\}\sqrt{\sigma^2+(r-\mu)^2}} \geq 0$$

Hence, $V(Q,r)$ is convex in r for any given Q. However, as in the classical order quantity/reorder point model, the joint convexity or quasi-convexity of $V(Q,r)$ in (Q,r) is not clear. The first order necessary conditions for the minimisation of $V(Q,r)$ yield:

$$Q^2\int_Q^\infty g(u)\mathrm{d}u + 2Q\int_0^Q ug(u)\mathrm{d}u - \int_0^Q u^2 g(u)\mathrm{d}u$$
$$-\frac{2D}{h}\left[A + pE(X-r)^+\right] = 0$$

and

$$\frac{r-\mu}{\sqrt{\sigma^2+(r-\mu)^2}} = 1 - \frac{2h}{Dp}\left\{Q\int_Q^\infty g(u)\mathrm{d}u + \int_0^Q ug(u)\mathrm{d}u\right\}$$

If the variable capacity is exponentially distributed with mean β, the optimal Q and r satisfy the following equations.

$$\begin{array}{l} 2\beta^2[exp(-Q/\beta)-1] + 2\beta Q - \frac{2DA + D\sqrt{\sigma^2+(r-\mu)^2}-(r-\mu)}{h} = 0 \\ \frac{r-\mu}{\sqrt{\sigma^2+(r-\mu)^2}} = 1 - \frac{2\beta h}{Dp}[1-exp(-Q/\beta)] \end{array} \tag{12}$$

5 Numerical examples and computational experiments

Example 1: We first consider a basic EOQ system with the following data: A = \$50 per order, h = \$5 per item per year, D = 1,000 items per year, and c = \$5 per item. The variable supplier capacity is exponentially distributed with a mean, β, of 100 items. The yield rate, R, is uniformly distributed as $U(0.8, 1)$.

Table 1 The effect of a variable supplier capacity (β)

β	Q^*	$V_R(Q^*)$
200	180.3	5,814.6
300	171.7	5,776.1
400	167.8	5,758.0
500	165.5	5,747.6
1,000	161.0	5,727.6
10,000	157.2	5,710.4

Table 2 The effect of the random yield parameter (R)

$U(a,b)$	Q^*	$V_R(Q^*)$
$U(0.8,1)$	210.8	5,952.5
$U(0.85,1)$	203.6	5,943.5
$U(0.9,1)$	196.7	5,935.3
$U(0.95,1)$	190.3	5,927.7
$U(0.99,1)$	185.3	5,922.1
$U(1,1)$	184.1	5,920.7

Table 3 The effect of the mean random yield (R)

$U(a,b)$	Q^*	$V_R(Q^*)$
$U(0.75,0.8)$	258.9	6,003.7
$U(0.8,0.85)$	237.8	5,980.3
$U(0.85,0.9)$	219.4	5,960.3
$U(0.9,0.95)$	203.8	5,942.9
$U(0.95,1)$	190.3	5,927.7

Table 4 The effect of the variance of the random yield (R)

$U(a,b)$	Q^*	$V_R(Q^*)$
$U(0.8,1)$	210.8	5,952.5
$U(0.85,0.95)$	211.2	5,951.5
$U(0.89,0.91)$	211.4	5,951.2

Table 5 The effect of the variable supplier capacity ($U(l,m)$)

Mean($U(l,m)$)	Q^*	$V_R(Q^*)$
200($U(0,400)$)	169.2	5,764.5
300($U(0,600)$)	164.5	5,743.3
400($U(0,800)$)	162.4	5,733.8
500($U(0,1{,}000)$)	161.2	5,728.4
1,000($U(0,2{,}000)$)	158.9	5,718.1
10,000($U(0,20{,}000)$)	157.0	5,709.4

First, we insert the above parameters into equation (5). Using a bisection algorithm, we can obtain the optimal order quantity, Q^*, as Q^* = 210.8. Substituting the value of Q^*, we calculate the expected cost per unit time $V_R(Q^*)$ as $V_R(Q^*)$ = \$5,952.5. If the supplier capacity is infinite ($\beta = \infty$), then Q^* = 156.8 and $V_R(Q^*)$ = \$5,708.6. In this case, the values of Q^* and $V_R(Q^*)$ are decreased by 25.6% and 4.1%, respectively.

To investigate the effect of the parameters, we compute $V_R(Q^*)$ using randomly generated problems within realistic ranges of parameters. First, we solve the cases when β= 200, 300, 400, 500, 1,000, and 10,000 (in this situation, the other parameters are the same). The computational results are summarised in Table 1. From Table 1, we see the impact of variable supplier capacity on Q^* and $V_R(Q^*)$. That is, an increase in supplier capacity, β, results in a decrease in Q^* and $V_R(Q^*)$. Second, we solve the case when the parameter of the random yield, R, varies. From Table 2, we see that Q^* and $V_R(Q^*)$ decrease as the random yield, R, increases. Finally, we analyse the effect of the mean and variance of R on Q^* and $V_R(Q^*)$. From Table 3, we see that Q^* and $V_R(Q^*)$ decrease as the mean of R increases. Note that the variances are the same for all cases in Table 3. From Table 4, we see that Q^* and $V_R(Q^*)$ decrease as the variance of R decreases. Note that we used the same mean for all cases in Table 4. We can see that the optimal order quantity and expected cost per unit decrease as the supplier capacity increases and the yield rate of item increases. The computational results are summarised in Table 5 for the case when the variable supplier capacity is uniformly distributed as $U(l, m)$ (in this situation, the other parameters are the same as in the previous example).

Example 2: We now consider a multi-item EOQ model. A small electronics company purchases three types of components. The management desires never to have an investment in these items in excess of \$10,000. The variable supplier capacity is exponentially distributed with a mean $\beta_1 = 100$ items, $\beta_2 = 158$ items, and $\beta_3 = 112$ items, respectively. No backorders are allowed. The pertinent data for each item are shown in Table 6. We determine the optimal lot size for each item.

Table 6 Data for example

	Item 1	*Item 2*	*Item 3*
Demand rate, D_i	1,000	1,000	2,000
Item cost, c_i	50	20	80
Set up cost, A_i	50	50	50
Holding cost, h_i	10	4	16

Step 1 Ignoring the budget constraint, the optimal order quantities are obtained by using equation (7) as $Q_1 = 119.8$, $Q_2 = 189.5$, and $Q_3 = 133.9$. Then, the total investment is \$11,948.9. This value exceeds the budgetary constraint (\$10,000). Thus, we go to Step 2.

Step 2 We select λ arbitrarily. Let $\lambda = 0.5$.

Step 3 Using equation (8), we compute Q_i. $Q_1 = 50.4$, $Q_2 = 79.7$, and $Q_3 = 56.3$.

Step 4 Then the total expected investment is \$6,773.0.

This value is less than the budget for the total investment. Thus, we decrease λ and go to Step 3. If we continue this procedure, we find the following optimal order quantities of $(Q_1^*, Q_2^*, Q_3^*) = (87.8, 138.9, 98.1)$ with the optimal Lagrangian multiplier, λ^*, being 0.1207 in this case. The computational results are summarised in Table 7 where:

$$TI = \sum_{i=1}^{n} c_i\{Q_i \int_{Q_i}^{\infty} g_i(u_i)\mathrm{d}u_i + \int_{0}^{Q_i} u_i g_i(u_i)\mathrm{d}u_i\}.$$

Table 7 Summary of the solution procedure

λ	Q_1^*	Q_2^*	Q_3^*	TI
0.0000	119.8	189.5	133.9	11,948.9
0.5000	50.4	79.7	56.3	6,773.0
0.0500	104.3	164.9	116.5	11,078.8
⋮	⋮	⋮	⋮	⋮
0.1207	87.8	138.9	98.1	10,000.0

To investigate the effect of the capacities, we solve the case when β_1, β_2 and β_3 vary (in this situation, the other parameters are the same). The results are summarised in Table 8.

Table 8 The effect of variable supplier capacities (β_1,β_2,β_3)

$(\beta_1, \beta_2, \beta_3)$	λ^*	Q_1^*	Q_2^*	Q_3^*
(100, 180, 150)	0.1389	84.4	128.8	87.6
(300, 180, 150)	0.1560	67.8	124.3	84.6
(300, 500, 150)	0.1637	66.7	105.1	83.3
(1,000, 1,000, 1,000)	0.1878	60.3	96.7	67.6

Example 3: We solve an example for (Q, r) model taken from Wang and Gerchak (1996a). Let A = \$50 per order, c = \$5 per item, h = \$2 per year per item, D = 200 items per year, and p = \$25 per item; The lead time demand is normally distributed with mean, μ = 100 items, and standard deviation, σ = 25 items. The variable supplier capacity is exponentially distributed with a mean, β = 100 items.

Using the line search algorithm, we obtain the optimal solution as follows: $Q^N = 129.6$ items and $r^N = 147.4$ items where N stands for the normal distribution. Then, the expected cost per unit time, $V^N(Q^N, r^N)$, is \$5,097.9. If we apply the DFA for forecasting the lead time demand, we obtain the optimal solution by using equation (12): $Q^W = 194.6$ items and $r^W = 164.0$ items where W stands for the worst distribution. If we use the quantity (Q^W, r^W) instead of (Q^N, r^N), the expected loss is equal to:

$$V^N(Q^W, r^W) - V^N(Q^N, r^N)$$

This is the largest amount that one has to pay for the knowledge of a distribution function. This quantity can be regarded as the *expected value of additional information* (EVAI) (Gallego and Moon, 1993). Since the value of $V^N(Q^W, r^W)$ is \$5,131.4, the EVAI is:

$$V^N(Q^W, r^W) - V^N(Q^N, r^N) = \$5{,}131.4 - \$5{,}097.9 = \$33.5$$

The ratio, $\frac{V^N(Q^W, r^W)}{V^N(Q^N, r^N)}$, is 1.0066.

We investigate the effects of the parameters $(\beta,\ \mu,\ \sigma)$ on the optimal results. First, we solve the cases when $\beta = 200, 300, 400, 500,$ and $1{,}000$ (in this situation, the other parameters are the same). The results are summarised in Table 9. From Table 9, we see that the values of all the parameters decrease as the value of β increases. Moreover, the ratios $(\frac{V^N(Q^W,r^W)}{V^N(Q^N,r^N)})$ are quite close to 1. Second, Table 10 shows the comparative results under several different parameters $(\mu,\ \sigma)$ of the lead time demand. Most ratios $(\frac{V^N(Q^W,r^W)}{V^N(Q^N,r^N)})$ are quite close to 1, which enables us to use the distribution free ordering rule in the absence of knowledge on the specific distribution. We know the following facts from Table 10. $V^N(Q^N, r^N)$, $V^N(Q^W, r^W)$ and $\frac{V^N(Q^W,r^W)}{V^N(Q^N,r^N)}$ are significantly affected by the standard deviation, σ. That is, the values of the above cost functions and the ratio decrease as σ decreases. These results imply that the uncertainty of demand significantly affects the ordering policy.

Table 9 The effect of variable supplier capacity (β)

β	Q^W	r^W	Q^N	r^N	$V^N(Q^N, r^N)$	$V^N(Q^W, r^W)$	$\frac{V^N(Q^W,r^W)}{V^N(Q^N,r^N)}$
200	177.3	153.5	119.3	145.0	5,090.7	5,114.1	1.0046
300	172.1	150.2	116.3	144.2	5,087.2	5,105.5	1.0036
400	169.7	148.6	114.9	143.8	5,084.5	5,102.0	1.0034
500	168.3	147.7	114.0	143.5	5,082.2	5,099.9	1.0035
1,000	165.5	145.9	112.4	143.1	5,072.3	5,096.0	1.0047

Table 10 The effect of the lead time demand (μ, σ)

(μ,σ)	Q^W	r^W	Q^N	r^N	$V^N(Q^N, r^N)$	$V^N(Q^W, r^W)$	$\frac{V^N(Q^W,r^W)}{V^N(Q^N,r^N)}$
(50,10)	150.1	77.0	123.7	69.04	5,041.0	5,057.1	1.0032
(50,50)	265.9	172.2	139.9	144.0	5,191.3	5,249.3	1.0112
(100,25)	194.6	164.0	129.6	147.4	5,097.9	5,131.4	1.0066
(150,10)	150.1	177.0	123.7	169.1	5,041.1	5,057.1	1.0032
(150,50)	265.8	272.3	139.9	244.0	5,191.2	5,249.5	1.0112

6 Concluding remarks

In this paper, we have presented models for the lot sizing problem with variable supplier capacity under an EOQ framework. Regarding defective items, if a random yield rate is not considered, our model would be the same as that of Wang and Gerchak (1996a). If variable supplier capacity is not considered, our model would be the same as that of Silver (1976). By considering multiple items, we see that the supplier capacity for an item affects the optimal order quantities of all items. We have obtained the following main results. The ordering policy (viz. order quantity and reorder point) and the expected cost per unit time are significantly affected by the uncertainty of supply. That is, variable supplier capacity is always found to be a cause of increases in the optimal lot size, as compared to the case of unlimited capacity. We have applied the DFA to the (Q, r) model with variable supplier capacity. Notwithstanding its simplicity and conservatism, the DFA has produced very nice results as can be observed from various computational experiments. We encourage the managers to use the DFA rather than spending their efforts to estimate the specific form of the distribution.

In this paper, we assume that the proportional cost is applied only to the actually executed production or delivered quantity. Even though this assumption is quite practical, some suppliers should pay the penalty if they cannot satisfy the order quantity of the major manufacturing company. Extending the current models by accommodating different cost structures might be an interesting research problem. Another interesting research issue is to quantify and/or bound the effect of the uncertainty in supply to the ordering policy.

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