INT. J. PROD. RES., 2004, VOL. 42, NO. 4, 677–691

Accounting for idle capacity cost in the scheduling of economic lot sizes

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This paper considers the issue of idle capacity cost in determining economic lot sizes. Two mathematical models are developed for the economic lot scheduling problem (ELSP). In Model I, the ELSP with fixed production rates is formulated under both the common cycle and time-varying lot sizes approaches. The associated constrained optimization problem in the time-varying lot sizes approach is reduced to solving a parametric quadratic programming problem. In Model II, the modified ELSP (or MELSP) is treated with variable production rates and unit production cost of each item as a function of its production rate. An upper bound and a lower bound on the MELSP are derived. Lot-sizing decisions of the proposed models are obtained and their dependencies on the idle capacity cost are examined with numerical examples.

1. Introduction

The classical economic lot scheduling problem (ELSP) assumes that a single facility is dedicated to the production of a family of products with the restriction that it can be used to produce only one product at a time. The demand and production rates of the products are deterministic and uniform. The objective is to determine a feasible production schedule which is repeatable over an infinite planning horizon so that the demands are met without stockouts in each cycle and the long-run average inventory cost is minimized.

Most of the studies in the ELSP literature assume that the production rates of the products are predetermined and inflexible. In order to reduce the effective production rates in the facility, Sheldon (1987) employed an idle time insertion strategy that divides the production runs into two parts. In the first part idle times are inserted to produce at the demand rate and in the second part no idle time is inserted to produce at the nominal rate. Inman and Jones (1989) studied the ELSP by slowing down the production rate uniformly over a run by inserting small and identical idle times between production of each consecutive unit. Silver (1990) showed that under certain circumstances significant cost savings can be achieved by slowing down the production rate of just one key item in the family. He used a common cycle interval to all the items to analyse the rigid case in which production rates cannot be changed during the production runs. Moon *et al.* (1991) generalized Silver's (1990) work by analysing the flexible case in which the production rates are controllable during the production run. They showed that the optimal production schedule would be the one in which the products are to be produced first to meet the demands and then to

Revision received November 2002.

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International Journal of Production Research ISSN 0020–7543 print/ISSN 1366–588X online © 2004 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/0020754031000164384

produce at the maximum rates, that is, larger idle time would be allowed initially and then none at all. Gallego (1993) reconsidered both the rigid and flexible cases but did not adopt the common cycle approach like Silver (1990). In the rigid case, he showed that it is optimal to reduce the rate of at most one item while minimizing a sharp lower bound on the average cost and the results of this lower bound solution can be utilized to obtain an optimal or near-optimal cyclic schedule in the time-varying lot sizes heuristic. In the flexible case, he suggested a cyclic schedule in which additional savings are possible by first producing at the demand rate and then resuming at the nominal rate.

The primary aim of all the above research is to slow down the production rates by inserting the idle time especially when the utilization of the facility is not high because reduced production rate can decrease accumulation of inventories which results in a decrease in inventory holding cost. But these works did ignore the likely effects of increased unit production costs resulting from the reduced production rates on the lot scheduling decision. Khouja (1997) provided an extension to the ELSP in which the production rates are treated as decision variables and the average unit cost of each item as a function of its production rate. He studied the problem using the common cycle approach. Moon and Christy (1998) extended the work of Silver (1990) by considering upper and lower limits of production rates and mold cost. Later, Khouja (1999) analysed an ELSP in which the production rates are controllable prior to the start of the production run and quality levels deteriorate with increased production rates and lot sizes.

In a single-machine multi-product situation, if adequate capacity is available to produce all the products (i.e. any left over or idle capacity is not needed to satisfy any other demand), then the cost of idle equipment and/or labour needs to be accounted for in the total cost of the ELSP in order to have an accurate analysis. The issue of idle capacity on lot-sizing decisions was studied by Eiamkanchanalai and Banerjee (1999) who developed a model for simultaneously determining the optimal run length and production rate for a single-product case. They assumed that the total idle capacity cost is linear in the idle time. When the capacity is relatively tight, they defined the cost of idle capacity as negative (benefit); otherwise, the cost is positive (penalty). In the present article, we extend the work of Eiamkanchanalai and Banerjee (1999) to a single-machine multi-product case and develop two mathematical models under two different scenarios. In Model I, the production rates of the products are assumed to be fixed and the lot scheduling problem is studied under both the common cycle and time-varying lot sizes approaches. The constrained minimization problem in the time-varying lot sizes approach is reduced to solving a parametric quadratic programming problem. Model II is concerned with the modified ELSP (or MELSP) in which the production rates are treated as decision variables and the unit production cost of each item as a function of its production rate. An upper bound and a lower bound on the MELSP are derived. Lot-sizing decisions of the proposed models are obtained and their dependencies on the idle capacity cost are examined with numerical examples.

2. Basic assumptions and notation

The following assumptions are made in developing the models:

- Several items compete for the use of a single machine and only one item can be produced at a time.

- Demands are deterministic and uniform.
- Production rates are fixed in Model I and flexible in Model II.
- Set-up costs and times are product dependent but independent of the production sequence.
- Shortages are not permitted.
- At time zero, there is just enough on-hand inventory of each product to satisfy
- demand until the first scheduled production of the product.

The following notation is used throughout the paper:

i: item index, $i = 1, 2, \ldots, m$
D_i: constant demand rate of item i
P_{1i}: (fixed) production rate of item i in Model I
P_{2i}: (flexible) production rate of item i in Model II
T_{1i}, T_{2i}: cycle lengths for item i
T_1, T_2: fundamental cycle lengths
A_i: fixed set-up cost for item i
s_i: known set-up time for item i
t_i: production run time for item i
r: inventory holding cost rate
C_i: unit production cost of item i in Model I
$C_i(P_{2i})$: unit production cost (which is a function of the production rate P_{2i}) of item i in Model II
P_{2i}^0: production rate that minimizes the unit production cost for item i in Model II
C_d: cost of idle or left-over capacity per unit idle time
U_i: upper bound of the production rate P_{2i}
TC_{2i}: total cost per unit time for item i in Model II
TC_1, TC_2: total costs per unit time for all items.

3. Model I: The ELSP with fixed production rates and idle capacity cost

We will develop the model incorporating the idle capacity cost but ignoring the production cost because every production schedule will have the same production cost over the infinite time horizon, namely the sum of C_iD_i over all i.

3.1. *Common cycle approach*

In this approach, the cycle times of all the products are assumed to be equal. The products are produced once in each cycle and between the times of production of any of the products every other product is produced (Hanssmann 1962). As the products are produced only once, the production sequence is not important. Figure 1 shows a common cycle for three products, assuming that the items are produced in the order 1-2-3.

The cost function of the ELSP can be obtained as

$$\mathrm{TC}_1(T_1) = \sum_{i=1}^{m}\left[\frac{A_i}{T_1} + \frac{1}{2}rC_iD_i\left(1 - \frac{D_i}{P_{1i}}\right)T_1\right] + C_d\left[1 - \sum_{i=1}^{m}\left(\frac{s_i}{T_1} + \frac{D_i}{P_{1i}}\right)\right]. \qquad (1)$$

Since $\left(1 - \sum_{i=1}^{m}(D_i/P_{1i})\right)$ is the proportion of time available for machine set-ups and $\left(\sum_{i=1}^{m}(s_i/T_1)\right)$ is the proportion of time actually needed for set-ups of all the products, therefore $\left[1 - \sum_{i=1}^{m}(s_i/T_i + D_i/P_{1i})\right]$ is the proportion of time the machine

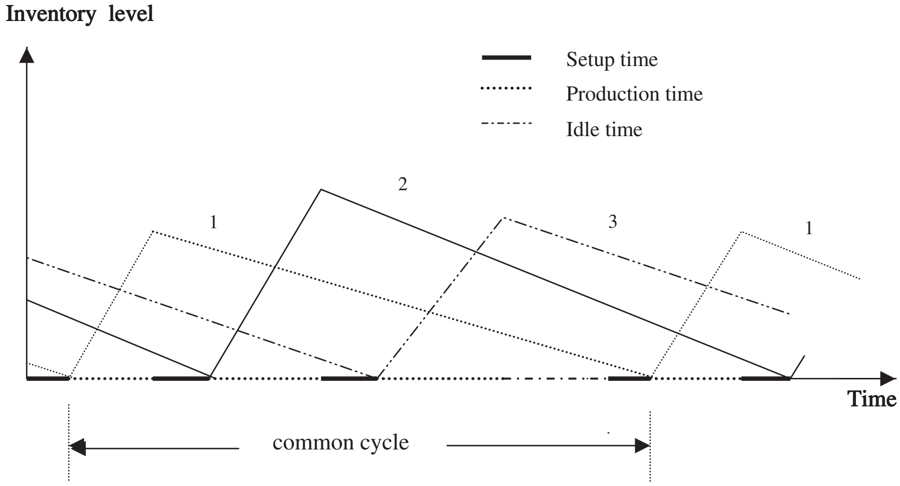

Figure 1. Production schedule for three items in the common cycle approach.

remains idle. Let $\rho_{1i}(=(D_i/P_{1i}))$ represent the machine utilization or load due to the product i. Then obviously $\rho_1 = \sum_{i=1}^{m} \rho_{1i} < 1$. The necessary condition for the minimum of $\mathrm{TC}_1(T_1)$ gives

$$T_1 = \sqrt{\frac{\sum_{i=1}^{m}(A_i - C_d s_i)}{\frac{1}{2} r \sum_{i=1}^{m} C_i D_i (1-\rho_{1i})}}.$$

This implies that $C_d < \sum_{i=1}^{m} A_i / \sum_{i=1}^{m} s_i$ for the existence of a real optimal point T_1 and in that case any local minimum is indeed the global minimum as $\mathrm{TC}_1(T_1)$ is convex when $C_d < \sum_{i=1}^{m} A_i / \sum_{i=1}^{m} s_i$. We will consider T_1 as the optimal cycle length provided it satisfies the following feasibility constraint:

$$T_1 \geq \sum_{i=1}^{m}\left(s_i + \frac{D_i T_1}{P_{1i}}\right) \tag{2}$$

that is, $T_1 \geq \sum_{i=1}^{m} s_i/\kappa = (T_1)_{\min}$ (say), where $\kappa = 1 - \rho_1$. Hence the optimal cycle length would be $\max\{T_1, (T_1)_{\min}\}$ provided a real T_1 exists for optimality.

3.2. *Time-varying lot-sizes approach*

In the time-varying lot-sizes approach, some items may be produced several times during a cycle and their production runs within a cycle may be different giving different lot sizes. The problem requires us to specify the production sequence first and then to determine the production times, idle times and the cycle time of the sequenced products. Dobson (1987) showed that given a production sequence $\boldsymbol{f}$, there always exists a feasible solution of the problem if and only if $\rho_1 < 1$. To obtain the production sequence $\boldsymbol{f}$, we use the bin-packing heuristic which will be outlined in the next section.

Let n be the number of positions in the production sequence $\boldsymbol{f} = (f^1, f^2, \ldots, f^n)$ where $f^j = i$, if the item i is produced in position j. Let $J_i = \{j : f^j = i\}$, production time vector $\boldsymbol{t} = (t^1, t^2, \ldots, t^n)$ and idle time vector $\boldsymbol{w} = (w^1, w^2, \ldots, w^n)$. We use the subscripts to refer to the data related to the ith item and superscripts to the data related to the item produced at the jth position in the sequence. Thus

$A^j = A_{f^j}, D^j = D_{f^j}$, etc. Let F be the set of all possible finite sequences of the products and L_k denote the positions in a given sequence from k up to but not including the position in the sequence where the product f^k is produced again. With these definitions and notation, the ELSP can be formulated as given below:

$$\inf_{f\in F}\mathrm{Min}_{\mathbf{t}\geq\mathbf{0},\,\mathbf{w}\geq\mathbf{0},\,T_1>0}\frac{1}{T_1}\left[\sum_{j=1}^{n}A^j+\frac{r}{2}\sum_{j=1}^{n}C^j\left(\frac{P_1^j}{D^j}-1\right)P_1^j(t^j)^2+C_d\sum_{j=1}^{n}w^j\right] \quad (3)$$

subject to

- sufficient production time for each product i to meet its demand over the cycle

$$\sum_{j\in J_i}P_{1i}t^j = D_iT_1, \quad i=1,2,\ldots,m \quad (4)$$

- sufficient production of product f^k to meet its demand until the next time the same product is produced again

$$\sum_{j\in L_k}\left(t^j+s^j+w^j\right)=\frac{P_1^kt^k}{D^k}, \quad k=1,2,\ldots,n \quad (5)$$

- fundamental cycle time

$$\sum_{j=1}^{n}\left(t^j+s^j+w^j\right)=T_1 \quad (6)$$

- non-negativity restrictions on the variables

$$\boldsymbol{t}\geq\mathbf{0}, \boldsymbol{w}\geq\mathbf{0}, T_1>0.$$

Equations (4) are redundant, since if we substitute (6) into (4) we obtain

$$\sum_{j\in J_i}\left(\frac{P_1^j}{D^j}-1\right)t^j-\sum_{j\notin J_i}t^j=\sum_{j=1}^{n}(s^j+w^j)$$

which is the sum of (5) over $k\in J_i$. We now define the vectors $\boldsymbol{a}=(A^j)_{j=1}^n$, $\boldsymbol{s}=(s^j)_{j=1}^n$. Let $w=\boldsymbol{e}'\boldsymbol{w}$ and $a=\boldsymbol{e}'\boldsymbol{a}$ where $\boldsymbol{e}$ denotes the n-vector of ones. Let

$$b_{jk}=\begin{cases}1, & \text{for } k=j,j+1,\ldots,\text{next}(j)\text{-}1\\ 0, & \text{otherwise}\end{cases}$$

where next(j) denotes the next position k in f such that $f^k=f^j$.

Let $B=(b_{jk})_{j,k=1}^n$. We define the diagonal $n\times n$ matrices $E=\mathrm{diag}(\rho_1^j)$, $H=\mathrm{diag}(rC^jP_1^j(1-\rho_1^j)/\rho_1^j)$. If $A=EB$ then the problem described in (3)–(6) can be stated as

$$\inf_{f\in F}\mathrm{Min}_{\mathbf{t}\geq\mathbf{0},\,\mathbf{w}\geq\mathbf{0},\,T_1>0}\frac{1}{T_1}\left(\frac{1}{2}t'Ht+C_dw+a\right)$$

subject to

$$\boldsymbol{t}=A(\boldsymbol{s}+\boldsymbol{t}+\boldsymbol{w})$$
$$\boldsymbol{e}'(\boldsymbol{s}+\boldsymbol{t}+\boldsymbol{w})=T_1$$
$$\boldsymbol{t},\boldsymbol{w}\geq 0, T_1>0.$$

The constraints $\boldsymbol{w}\geq\mathbf{0}$ and $\boldsymbol{t}=A(\boldsymbol{s}+\boldsymbol{t}+\boldsymbol{w})$ imply that

$$\boldsymbol{t} = (I - A)^{-1} A(\boldsymbol{s} + \boldsymbol{w}) \geq \boldsymbol{0}$$

as $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$ and all the entries of A are non-negative. So the constraints $\boldsymbol{t} \geq \boldsymbol{0}$ are redundant. Our problem thus reduces to determine T_1 which minimizes

$$Z(T_1) = \frac{1}{T_1}[z(T_1) + a]$$

where

$$z(T_1) = \min\left[\left(\frac{1}{2}\boldsymbol{t}' H \boldsymbol{t}\right) + C_d w\right] \tag{7}$$

subject to

$$\boldsymbol{t} = A(\boldsymbol{s} + \boldsymbol{t} + \boldsymbol{w})$$
$$\boldsymbol{e}'(\boldsymbol{s} + \boldsymbol{t} + \boldsymbol{w}) = T_1$$
$$\boldsymbol{w} \geq \boldsymbol{0},\ T_1 > 0.$$

Thus we are now left to solve the above parametric quadratic programming problem in which the parameter T_1 appears in the problem linearly. Any pivotal-based technique can be applied for this purpose. Zipkin (1991) applied a complementary pivotal algorithm to solve such a problem. There is also commercial software, e.g., LINDO (Schrage 1991), GINO (Liebman *et al.* 1986) and Optimization Subroutine Library (OSL) IBM (1991) which can be applied to solve the problem.

3.3. *Algorithm to find the production sequence* $\boldsymbol{f}$

Step 1. Determine the relative production frequencies x_i from the lower bound solution (to be discussed in the next section) by the following relation:

$$x_i = \frac{\text{Max}_i\{T_{1i}^*\}}{T_{1i}^*}, \quad i = 1, 2, \ldots, m.$$

Step 2. Round off the relative production frequencies to power-of-two integers y_i where

$$y_i = 2^p \quad \text{if} \quad x_i \epsilon \left[\frac{1}{\sqrt{2}}2^p, \sqrt{2}2^p\right], \quad p = 0, 1, \ldots$$

Roundy (1989) showed that the additional costs due to the conversion of the real values of the production frequencies to power-of-two integers do not exceed 6%.

Step 3. Using the frequencies y_i, allocate the items in b bins where $b = \max_{1 \leq i \leq m} y_i$, with the aim of spreading them out as evenly as possible. While assigning the items to bins, a variation of the longest processing time (LPT) rule can be used in which the items are to be ordered lexicographically by (y_i, v_i), v_i being the estimated processing time of item i. By minimizing the maximum height of the bins, an efficient production sequence $\boldsymbol{f}$ can be determined.

Remarks: The above heuristic (which may be called the modified Dobson (1987) heuristic) is not the only heuristic to find the production sequence $\boldsymbol{f}$. Recently, Moon *et al.* (2002) successfully implemented genetic algorithms (GAs) to find a production sequence where they rounded the relative production frequencies to nearest integers to use a genetic scheme.

3.4. *A lower bound on cost*

A lower bound on the total cost of the problem can be found by considering the products as if they are scheduled on m facilities rather than on a single one. This leads to finding the individual cycle times, or equivalently, the economic lot sizes as in the classical deterministic inventory problem. Since no method of solving for an optimal feasible schedule exists, the purpose of developing such a model is to compare the non-optimal feasible solutions obtained previously in sections 3.1. and 3.2. with a lower bound on the total cost.

Problem LB

$$\text{Minimize}_{T_{11}, T_{12},\ldots, T_{1m}}\left[\sum_{i=1}^{m}\left\{\frac{A_i}{T_{1i}}+\frac{1}{2}rC_iD_i(1-\rho_{1i})T_{1i}\right\} + C_d\left\{1-\sum_{i=1}^{m}\left(\rho_{1i}+\frac{s_i}{T_{1i}}\right)\right\}\right] \tag{8}$$

subject to

$$\sum_{i=1}^{m}\frac{s_i}{T_{1i}} \leq 1-\rho_1 \tag{9}$$

$$T_{1i} \geq 0, \quad \forall i = 1,2,\ldots,m. \tag{10}$$

Proposition 1. There exists a unique optimal point for the lower bound model provided $C_d < A_i/s_i$ $\forall i = 1,2,\ldots,m$.

Proof: The Hessian matrix of the associated objective function (8) is positive definite when $C_d < A_i/s_i$ $\forall i$. The proof follows as the constraint set is convex in the T_{1i}'s.

The optimal points of the lower bound model are, therefore, those points which satisfy the Karush–Kuhn–Tucker (KKT) conditions

$$A_i - \frac{1}{2}rC_iD_i(1-\rho_{1i})T_{1i}^2 + (\lambda - C_d)s_i = 0, \quad i = 1,2,\ldots,m \tag{11}$$

$$\lambda\left[1-\rho_1-\sum_{i=1}^{m}\frac{s_i}{T_{1i}}\right] = 0, \tag{12}$$

$\lambda \geq 0$ complementary slackness with $\sum_{i=1}^{m}(s_i/T_{1i}) \leq 1-\rho_1$.

The above conditions are derived assuming that the T_{1i}'s are non-trivial. Equation (11) yields

$$T_{1i} = \sqrt{\frac{A_i - C_d s_i + \lambda s_i}{\frac{1}{2}rC_iD_i(1-\rho_{1i})}}, \quad i = 1,2,\ldots,m. \tag{13}$$

Proposition 2. There exists a unique value of λ for the optimal T_{1i}'s.

Proof: If the values of T_{1i} $(i = 1,2,\ldots,m)$ for $\lambda = 0$ in (13) satisfy the capacity constraint (9), then they are optimal. Otherwise, if possible, let there exist two values of λ, say $\lambda_1, \lambda_2 (> 0)$, which satisfy equations (11) and (12). If $\lambda_1 > \lambda_2$ then we have from (13),

$$(T_{1i})_{\lambda_1} > (T_{1i})_{\lambda_2} \quad \forall i = 1,2,\ldots,m. \tag{14}$$

Since $\lambda_1, \lambda_2 > 0$, the capacity constraint is binding and hence

$$\sum_{i=1}^{m} \frac{s_i}{(T_{1i})_{\lambda_1}} = \sum_{i=1}^{m} \frac{s_i}{(T_{1i})_{\lambda_2}} = 1 - \rho_1.$$

This contradicts the relation (14). Hence, a unique value of λ gives the optimal T_{1i}'s. We can, therefore, apply a simple line search technique on λ to find the optimal values of the T_{1i}'s ($i = 1, 2, \ldots, m$) that determine the minimum total cost per unit time from (8).

4. Model II: The ELSP with variable production rates and idle capacity cost

In many real production situations, production capacity of the machine can be changed easily within its designed limit. If the production rates are considered as decision variables then the unit production cost of each item becomes a function of its production rate. In this section, we consider the ELSP with variable production rates and call it the modified ELSP or MELSP. We first solve the problem ignoring the synchronization constraint which means that no two items can be produced at the same time. The problem can be formulated as

$$\begin{aligned} \text{Minimize } \mathrm{TC}_2 &= \sum_{i=1}^{m} \mathrm{TC}_{2i} \\ &= \sum_{i=1}^{m} \left[\frac{A_i}{T_{2i}} + \frac{r}{2} D_i T_{2i}(1 - \rho_{2i}) C_i(P_{2i}) + D_i C_i(P_{2i}) \right] \\ &\quad + C_d \left[1 - \sum_{i=1}^{m} \left(\rho_{2i} + \frac{s_i}{T_{2i}} \right) \right], \quad \rho_{2i} = D_i / P_{2i} \end{aligned} \tag{15}$$

subject to the constraints

$$\sum_{i=1}^{m} \left(\rho_{2i} + \frac{s_i}{T_{2i}} \right) \le 1 \tag{16}$$

$$D_i \le P_{2i} \le U_i, \quad i = 1, 2, \ldots, m \tag{17}$$

$$T_{2i} \ge 0, \quad i = 1, 2, \ldots, m. \tag{18}$$

The assumption of P_{2i}'s as decision variables makes it hard to prove the convexity of the objective function (15). However, if $(\overline{P}_{21}, \overline{P}_{22}, \ldots, \overline{P}_{2m}; \overline{T}_{21}, \overline{T}_{22}, \ldots, \overline{T}_{2m})$ is an optimal solution to the optimization problem (15)–(18), then there must exist multipliers $\lambda; \lambda_1, \lambda_2, \ldots, \lambda_m; \mu_1, \mu_2, \ldots, \mu_m; \eta_1, \eta_2, \ldots, \eta_m$ satisfying the following Karush–Kuhn–Tucker (KKT) conditions:

$$\left[-\frac{A_i}{\overline{T}_{2i}^2} + \frac{r}{2} D_i (1 - \overline{\rho}_{2i}) C_i(\overline{P}_{2i}) + C_d \frac{s_i}{\overline{T}_{2i}^2} \right] - \frac{\lambda s_i}{\overline{T}_{2i}^2} - \eta_i = 0, \quad i = 1, 2, \ldots, m \tag{19}$$

$$\left[\frac{r}{2}\overline{T}_{2i}\overline{\rho}_{2i}^2 C_i(\overline{P}_{2i}) + D_i C_i'(\overline{P}_{2i})\left\{1 + \frac{r}{2}\overline{T}_{2i}(1-\overline{\rho}_{2i})\right\} + C_d\frac{D_i}{\overline{P}_{2i}^2}\right]$$

$$-\frac{\lambda D_i}{\overline{P}_{2i}^2} - \lambda_i + \mu_i = 0, \quad i = 1, 2, \ldots, m \tag{20}$$

$$\lambda\left[1 - \sum_{i=1}^{m}\left(\overline{\rho}_{2i} + \frac{s_i}{\overline{T}_{2i}}\right)\right] = 0 \tag{21}$$

$$\lambda_i\left[\overline{P}_{2i} - D_i\right] = 0, \quad i = 1, 2, \ldots, m \tag{22}$$

$$\mu_i\left[U_i - \overline{P}_{2i}\right] = 0, \quad i = 1, 2, \ldots, m \tag{23}$$

$$\eta_i T_{2i} = 0, \quad i = 1, 2, \ldots, m \tag{24}$$

where $\lambda \geq 0$; $\overline{\rho}_{2i} = (D_i/\overline{P}_{2i})$, $\lambda_i \geq 0$, $\mu_i \geq 0, \eta_i \geq 0$, $\forall i = 1, 2, \ldots, m$.

If $\overline{P}_{2i} = D_i \forall i = 1, 2, \ldots, m$ then from (19) we get, $A_i - C_d s_i + \lambda s_i + \eta_i \overline{T}_{2i}^2 = 0$ which is impossible when $C_d < A_i/s_i$. Therefore, $\overline{P}_{2i} > D_i \forall i$ and this gives $\lambda_i = 0$ from (22), for $i = 1, 2, \ldots, m$. Moreover, for non-trivial $\overline{T}_{2i}$'s, equations (24) implies that $\eta_i = 0 \ \forall \ i$. Thus the KKT conditions (19)–(24) reduce to (25)–(28) as given below:

$$A_i - \frac{r}{2}D_i(1-\overline{\rho}_{2i})\overline{T}_{2i}^2 C_i(\overline{P}_{2i}) + (\lambda - C_d)s_i = 0 \tag{25}$$

$$\frac{r}{2}\overline{T}_{2i}D_i^2 C_i(\overline{P}_{2i}) + D_i\overline{P}_{2i}^2 C_i'(\overline{P}_{2i})\left\{1 + \frac{r}{2}\overline{T}_{2i}(1-\overline{\rho}_{2i})\right\}$$

$$+ (C_d - \lambda)D_i + \mu_i\overline{P}_{2i}^2 = 0, \quad i = 1, 2, \ldots, m \tag{26}$$

$$\lambda\left[1 - \sum_{i=1}^{m}\left(\overline{\rho}_{2i} + \frac{s_i}{\overline{T}_{2i}}\right)\right] = 0 \tag{27}$$

$$\mu_i\left[U_i - \overline{P}_{2i}\right] = 0, \quad i = 1, 2, \ldots, m$$

$$\lambda \geq 0; \mu_i \geq 0, \qquad \forall i = 1, 2, \ldots, m. \tag{28}$$

If $\overline{P}_{2i} < U_i$ then from (28) $\mu_i = 0 \ \forall \ i$ and from (26) we have

$$\frac{r}{2}\overline{T}_{2i}D_i C_i(\overline{P}_{2i}) + \overline{P}_{2i}^2 C_i'(\overline{P}_{2i})\left\{1 + \frac{r}{2}\overline{T}_{2i}(1-\overline{\rho}_{2i})\right\} + C_d - \lambda = 0, \quad i = 1, 2, \ldots, m.$$

Thus for $\lambda \geq 0$, we have to determine the optimal point $(\overline{P}_{21}, \overline{P}_{22}, \ldots, \overline{P}_{2m}; \overline{T}_{21}, \overline{T}_{22}, \ldots, \overline{T}_{2m})$ satisfying the equations

$$A_i - \frac{r}{2}D_i(1-\overline{\rho}_{2i})\overline{T}_{2i}^2 C_i(\overline{P}_{2i}) + (\lambda - C_d)s_i = 0, \quad i = 1, 2, ..., m$$

$$\frac{r}{2}\overline{T}_{2i}D_i C_i(\overline{P}_{2i}) + \overline{P}_{2i}^2 C_i'(\overline{P}_{2i})\left\{1 + \frac{r}{2}\overline{T}_{2i}(1-\overline{\rho}_{2i})\right\} + C_d - \lambda = 0, \quad i = 1, 2, \ldots, m$$

$$\lambda\left[1 - \sum_{i=1}^{m}\left(\overline{\rho}_{2i} + \frac{s_i}{\overline{T}_{2i}}\right)\right] = 0.$$

Similarly, if $\overline{P}_{2i} = U_i,\ i = 1, 2, \ldots, m,$ then for $\lambda \geq 0,$ the $\overline{T}_{2i}$'s can be obtained from the following system of equations

$$A_i - \frac{r}{2} D_i \left(1 - \frac{D_i}{U_i}\right) \overline{T}_{2i}^2 C_i(\overline{P}_{2i}) + (\lambda - C_d) s_i = 0, \quad i = 1, 2, \ldots, m$$

$$\lambda \left[1 - \sum_{i=1}^{m} \left(\frac{D_i}{U_i} + \frac{s_i}{\overline{T}_{2i}}\right)\right] = 0, \quad i = 1, 2, \ldots, m.$$

4.1. *An upper bound on the MELSP*

The total cost in the common cycle solution is actually an upper bound on the MELSP. In the common cycle approach, the objective function which is to be minimized is given by

$$\mathrm{Min}\mathrm{TC}_2(T_2; P_{21}, P_{22}, \ldots, P_{2m}) = \sum_{i=1}^{m} \left[\frac{A_i}{T_2} + \frac{r}{2} T_2 D_i (1 - \rho_{2i}) C_i(P_{2i}) + D_i C_i(P_{2i})\right] + C_d \left[1 - \sum_{i=1}^{m} \left(\rho_{2i} + \frac{s_i}{T_2}\right)\right] \tag{29}$$

subject to

$$D_i \leq P_{2i} \leq U_i, \quad i = 1, 2, \ldots, m \tag{30}$$

$$T_2 \geq 0. \tag{31}$$

For non-trivial T_2, the KKT conditions for the minimum of TC_2 give

$$\sum_{i=1}^{m} A_i - \frac{r}{2} T_2^2 \sum_{i=1}^{m} D_i (1 - \rho_{2i}) C_i(P_{2i}) - C_d \sum_{i=1}^{m} s_i = 0 \tag{32}$$

$$\frac{r}{2} T_2 \left[D_i C_i(P_{2i}) + (P_{2i} - D_i) P_{2i} C_i'(P_{2i})\right] + P_{2i}^2 C_i'(P_{2i}) + C_d + \zeta_i \frac{P_{2i}^2}{D_i} = 0 \tag{33}$$

where ζ_i (≥ 0) are the Lagrange multipliers associated with the constraints $P_{2i} \leq U_i\ \forall\ i$. Arguing as before it can be shown that $D_i < P_{2i} \leq U_i\ \forall\ i = 1, 2, \ldots, m$. If equations (32) and (33) provide a solution $(T_2^*; P_{21}^*, P_{22}^*, \ldots, P_{2m}^*)$ then T_2^* can be taken as the optimal cycle length if

$$T_2^* \geq \frac{\sum_{i=1}^{m} s_i}{1 - \sum_{i=1}^{m} (D_i / P_{2i}^*)} \equiv (T_2)_{\min} \text{ (say)}.$$

If the cost function $\mathrm{TC}_2(T_2; P_{21}, P_{22}, \ldots, P_{2m})$ is convex then the optimal cycle length would be equal to $\max\{T_2^*, (T_2)_{\min}\}$.

5. Numerical examples

Example 1. For numerical study we consider a five-item batch production system. The data for the items taken from Banerjee *et al.* (1996) are shown in table 1.
The common cycle solutions of the ELSP for different values of the idle capacity cost C_d are presented in table 2 and the lower bounds of the corresponding total costs in table 3. It is found that for the idle capacity cost structure given in table 2 the left-over capacity in the common cycle approach is about 4%.

Product i	D_i (units/year)	P_{1i} (units/year)	s_i (hours)	A_i ($)	C_i ($)
1	18 050	153 120	4	400	275
2	34 026	153 120	6	600	350
3	35 980	153 120	10	1000	366
4	13 404	153 120	8	800	250
5	24 576	153 120	12	1200	250

Table 1. Data for the numerical example 1 ($r = 0.24$).

C_d ($/year)	T_1^* (year)	TC1 ($/year)
0	0.032 31	247 604
1000	0.032 29	247 648
5000	0.032 22	247 822
10 000	0.032 14	248 039
50 000	0.031 43	249 704

Table 2. Common cycle solutions for different values of C_d.

C_d ($/year)	T_{11} (year)	T_{12} (year)	T_{13} (year)	T_{14} (year)	T_{15} (year)	Lower bound ($/year)
0	0.027 59	0.023 23	0.028 76	0.046 69	0.044 03	238 955
1000	0.027 58	0.023 22	0.028 74	0.046 68	0.044 01	239 001
5000	0.027 51	0.023 17	0.028 68	0.046 60	0.043 91	239 185
10 000	0.027 43	0.023 10	0.028 60	0.046 51	0.043 78	239 413
50 000	0.026 79	0.022 56	0.027 93	0.045 74	0.042 76	241 172

Table 3. Lower bound solutions for different values of C_d.

Following the algorithm outlined in section 3.3, the production sequence of the model can be determined as $\boldsymbol{f} = \{3, 2, 1, 5, 3, 2, 1, 4\}$. To find the solution of the parametric quadratic programming problem (7), we use the commercial software package LINDO. The objective function of the quadratic programming problem is not directly input to LINDO as it requires all rows to be linear. The input procedure in LINDO is LP based and requires an objective function simply to identify the order of the variables which in turn determines the correspondence between variables and rows, see Schrage (1991) for the programming technique in detail. We take $C_d = 10\,000$ and search for the minimum value of $z(T_1)$ starting from $T_1 = (T_1)_{\min} + \epsilon$. After a few iterations we find the following required results:

$\mathbf{t}$ = (0.006 28, 0.005 76, 0.003 01, 0.008 02, 0.005 47, 0.005 35, 0.002 89, 0.004 38) year
$\mathbf{w}$ = (0, 0, 0, 0, 0, 0, 0, 0.002 26) year
T_1 = 0.049 98 year or 18.24 days
$Z(T_1)$ = $241 076/year.

Applying the same procedure for C_d = 0, 1000, 5000 and 50 000, we obtain the time-varying lot-sizes solutions of the ELSP as shown in table 4. In each case we observe that the idle time is only allowed at the end of the production of the last item in the sequence and the idle capacity proportion is nearly 0.002. So the capacity utilization

C_d ($/year)	T_1 (year)	$Z(T_1)$ ($/year)
0	0.050 80	240 623
1000	0.050 68	240 688
5000	0.050 32	240 850
10 000	0.049 98	241 076
50 000	0.048 78	242 812

Table 4. Time-varying lot-sizes solution for different values of C_d.

Product i	g_i	b_i	r_i
1	$(15\ 312)^2 \times 0.2694$	0.002 694	−550
2	$(15\ 312)^2 \times 0.3429$	0.003 429	−700
3	$(15\ 312)^2 \times 0.3585$	0.003 585	−732
4	$(15\ 312)^2 \times 0.2449$	0.002 449	−500
5	$(15\ 312)^2 \times 0.2449$	0.002 449	−500

Table 5. Parameter values of the unit production cost function.

in the time-varying lot-sizes approach is higher than the capacity utilization in the common cycle approach. The annual average total cost increases with the idle capacity cost in both the approaches. But the annual average total cost in the time-varying lot-sizes approach always remains less than that of the common cycle approach, see tables 2 and 4. A comparison of the results given in tables 2, 3 and 4 indicates that the annual total cost in the common cycle approach is above the lower bound by more than 3% whereas in the time-varying lot-sizes approach, it is only less than 0.7%.

Recall that in Model II the production rates are taken as decision variables and the unit production cost of the individual item as a function of its production rate. To obtain the numerical solution of the MELSP we consider the following unit production cost function which was suggested by Khouja (1997).

$$C_i(P_{2i}) = r_i + \frac{g_i}{P_{2i}} + b_i P_{2i}, \quad \forall i = 1, 2, \ldots, 5$$

r_i, g_i and b_i all being real numbers to be chosen to provide the best fit for the estimated unit production cost function. Explanations of the terms forming the above unit production cost function are given in Khouja (1997). Each production cost function $C_i(P_{2i})$ has a unique minimum at $P_{2i}^0 = \sqrt{g_i/b_i}$. The values of the parameters involved in $C_i(P_{2i})$ are given in table 5. These parameters are constructed so that the nominal production rates are $P_{2i}^0 = P_{1i}$, $\forall\ i = 1, 2, \ldots, 5$, the minimum unit cost is the same as C_i, $i = 1, 2, \ldots, 5$ and a 20% increase in P_{2i}^o results in a 5% increase in unit production cost C_i. We also assume that $U_i = 155\,000$ units/year, for $i = 1, 2, \ldots, 5$.

Comparing the results given in tables 2 and 6 we see that consideration of the flexible production rates in the ELSP does not improve the common cycle solution and the lower bound significantly. So for the current data set the time-varying lot

C_d ($/year)	Common cycle solution T_2(year)	$UB^{\#}$ ($/year)	Lower bound $LB^{\#}$ ($/year)
0	0.032 31	247 590.76	238 940.87
1000	0.032 29	247 634.16	238 986.67
5000	0.032 22	247 806.25	239 169.04
10 000	0.032 14	248 021.05	239 395.11
50 000	0.031 43	249 657.04	241 126.00

Table 6. Upper and lower bounds of the MELSP $UB^{\#}/LB^{\#}$: Upper bound/lower bound excluding the production cost.

Product i	D_i (units/day)	P_{1i} (units/day)	s_i (hour)	A_i ($)	C_i ($)
1	100	30 000	1	20	0.0065
2	100	8000	1	80	0.1775
3	200	9500	2	120	0.1275
4	400	7500	1	80	0.1000
5	20	2000	4	440	2.7850
6	20	6000	2	180	0.2675
7	6	2400	8	120	1.5000
8	85	1300	4	720	5.9000
9	85	2000	6	800	0.9000
10	100	15 000	1	20	0.0400

Table 7. Data for the numerical example 2 ($r = 0.2$).

C_d	Model I TC_1 ($/day)	Model I $Z(T_1)$ ($/day)	Model I LB_1 ($/day)	Model II TC_2 ($/day)	Model II LB_2 ($/day)
0	847.75 (0.574 48)	767.65	760.40 (0.640 12)	847.59* (0.573 25)	760.30* (0.624 46)
50	876.41 (0.571 96)	798.20	791.59 (0.636 55)	876.08* (0.568 97)	791.33* (0.619 02)
150	933.34 (0.566 62)	859.13	853.31 (0.626 90)	932.19* (0.559 72)	852.27* (0.605 22)
250	989.72 (0.560 85)	919.54	913.70 (0.609 06)	986.96* (0.549 28)	911.13* (0.582 373)
350	1047.49 (0.554 58)	979.55	970.28 (0.506 57)	1039.98* (0.536 67)	965.01* (0.472 10)

*indicates the total cost per day excluding the production cost.

Table 8. Computaional results of example 2. Figures in parentheses indicate the proportion of idle capacity. LB_1 = a lower bound on the ELSP, LB_2 = a lower bound on the MELSP.

sizes solution of the ELSP can be regarded a good approximate solution to the MELSP even though we are not able to find the time-varying lot sizes solution of the MELSP. To study further the impact of left-over capacity on the ELSP and MELSP we consider the following 10-item example which is based on Bomberger's (1966) problem.

Example 2. The data for a 10-item problem are shown in table 7 and the computational results in table 8. Similar to example 1, to estimate the values of the parameters involved in the unit production cost function in Model II we assume that a 20% increase in production rate results in a 5% increase in the unit production cost.

From the numerical results of example 2 we observe that

(i) the optimal production rates which minimize the average total cost in Model II are slightly reduced from the fixed production rates of Model I;
(ii) the impact of idle capacity cost on the lot scheduling decision is prominent for higher values of C_d;
(iii) the improvements of the upper and lower bounds on the MELSP in comparison to those on the ELSP are low; and
(iv) the total cost per day in the common cycle approach varies over the lower bound by 8–11% whereas this variation in the time-varying lot-sizes approach is less than 1%.

6. Conclusion

Capacity utilization is an important factor in today's competitive manufacturing environment. A manufacturing setting with low-capacity utilization may look better equipped to handle demand variability than a facility with high capacity utilization. The analysis of such a manufacturing system would be incomplete if the cost of idle capacity (idle equipment and/or labour) is significant and is not included in the model. In this paper we have studied the economic lot scheduling problem with both fixed and variable production rates under the framework of an idle capacity cost structure. We have assumed that the idle capacity cost is linearly related to the left-over capacity, though in practice it is hard to get an exact relationship. The ELSP with fixed production rates is formulated in Model I under both the common cycle and the time-varying lot-sizes approaches. The constrained minimization problem in the time-varying lot-sizes approach is solved by reducing it to a parametric quadratic programming problem. The modified ELSP (or MELSP) with variable production rates is formulated in Model II. The complexities in solving nonlinear programming problem in the time-varying lot-sizes approach restrict us to deal the MELSP under the common cycle approach only. It is evident from the numerical study that the cost of left-over capacity (idle equipment and/or labour) has a significant impact on the economic lot scheduling decision.

Acknowledgements

The research of Ilkyeong Moon has been supported by a Pusan National University Research Grant.

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