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Theory and Methodology

The multi-item single period problem with an initial stock of convertible units

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Abstract

This paper deals with the situation of a number of end items, each facing uncertain demand in a single period of interest. Besides being able to purchase units of the end items there is also available a stock of units that can be converted into end items but at unit costs that depend on the specific end item. Efficient solution procedures are presented for two situations: (i) where the end item demand distributions are assumed known (illustrated for the case of normally distributed demand) and (ii) a distribution free approach where only the first two moments of the distributions are assumed known. Computational results for a set of problems are presented. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we consider the situation of a number of end items facing uncertain demand over a single period (the multi-item newsvendor situation where purchase decisions must be made for each end item) but where there is a stock of convertible units that can be transformed into any of the end items at unit costs that depend upon the specific end items. This type of situation was first observed by one of the authors in a consulting assignment for a telecommunications organization where customers returned used telephone units that could be converted into other usable units (through repairs, adding a different colored plastic cover, etc.). In this connection Copacino [4] reports "Refurbishing of products: Products such as vending machines, computers, telephone equipment, and circuit boards are repaired and placed in inventory for resale. Many companies are moving to hold more unrepaired equipment in inventory and refurbishing to

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order in a just-in-time fashion''. Another less obvious application is in a supply chain context where partially processed items can be converted to different end items, e.g. personal computer printers for sale in different countries (see [7,13]). Part of the manufacturing is done centrally and units are shipped to various destinations where localized finishing (e.g. addition of special power source, user manual, etc.) is done. Partially processed units at a particular location could be processed further there or transshipped to other locations for completion. Also, in a supply chain context, some items can be held centrally and then converted (by transportation) to finished items at different locations. The work of Cheung and Powell [3] is in this spirit. Of course, a single period model is likely to be an approximation (and building block for more complicated modeling) in some of the above illustrated situations where there is on-going demand.

The convertible situation is clearly related to the contexts of service parts and repairable items for which there is a substantial literature. References, that include broad surveys of modeling efforts in these contexts, include Brown [2], Diks et al. [5], Mabini and Gelders [14], Nahmias [19], Sherbrooke [22], Silver et al. [24], and Verrijdt [26]. In addition, there is a connection with the so-called commonality problem where there are a number of end items each having a number of components and some of the components are common to more than one end item. In a sense our problem can be viewed as an extension of a special case of the commonality situation, namely where there is a single component that is common to all end items, but which can also be purchased externally. Some of the more recent references on the commonality problem include Bagchi and Gutierrez [1], Eynan [6], and Jönsson et al. [12]. Silver et al. [24] provide a summary of the associated literature.

In the next section we present the notation, assumptions, and basic mathematical model for the case where we have a *known* distribution of demand for each of the end items. Properties of the optimal solution (number of units to be converted to each end item and number of units of each end item that are purchased) are presented as well as an efficient, associated algorithm. Then Section 3 deals with the distribution free situation where all we assume are known are the means and variances of the demand distributions. Additional related references are provided at that stage. Section 4 presents the results associated with a set of test problems. Concluding remarks are provided in Section 5.

2. Basic model

2.1. Notation

The notation to be used is as follows:

j	index for end items ($j = 1, 2, \ldots, J$)
c_j	unit conversion cost from the convertible item to end item j, in dollars/unit
g_0	salvage value of convertible item, in dollars/unit
g_j	salvage value of end item j, in dollars/unit
v_j	purchase cost of end item j, in dollars/unit
B_j	penalty for not satisfying demand of end item j, in dollars/unit
I_j	initial inventory of end item j, in units
D_j	demand for end item j during the period, in units
$f_j(D_j)$	probability density function of demand for end item j
$F_j(D_j)$	cumulative probability distribution of demand for end item j
μ_j	expected value of demand for end item j
σ_j	standard deviation of demand for end item j
N	available number of units of the convertible item, in units

R_j number of convertible units that are converted into end item j (decision variables)
Q_j number of units of end item j that are purchased (decision variables)
x^+ $\max\{x, 0\}$

2.2. Assumptions

We use the following assumptions:

(i) We consider a single period model with stochastic demand in which demand can be satisfied from two methods: conversion and purchasing.

(ii) The demand for item j has a probability density function with known mean and standard deviation.

(iii) There exists an initial inventory for each end item. This inventory is less than the optimal unconstrained order-up-to level based on the item's purchase cost. (This optimal value will be defined later.) Thus, if the conversion option was not available, some units of this item would definitely be purchased. (In the single period context of this paper one would expect to have little, if any, initial inventory of the item available, hence the reasonableness of this assumption.)

(iv) The unit salvage value of each item after conversion is less than its unit conversion cost. Moreover, the unit salvage value of convertible items g_0 is less than $v_j - c_j$ for all j.

(v) The conversion and purchase decisions have to be made simultaneously (i.e. the associated lead times are the same).

Regarding assumption (iv), if there is an item whose v_j value is less than or equal to $c_j + g_0$, there is no need to convert units to that item. We can purchase the item using its order-up-to level computed independently of other items. Thus, we can eliminate the item from further consideration.

2.3. Modeling and algorithm

We can interpret the problem as follows: A service department (or purchasing department), which is in charge of providing items to other departments or factories, has N units of an item that are not directly usable. The unit salvage value is g_0. Each unit can be converted to an end item j ($j = 1, 2, \ldots, J$) at a cost of c_j. The end item can also be purchased at a unit cost of v_j. There is a cost B_j per unit of demand for item j not satisfied. Since it takes some time to convert or purchase items, we need to decide how many units should be converted and how many purchased in advance of knowing demands. This problem is similar to the classical newsvendor problem [24]. However, there is the added opportunity for conversion.

The expected cost can be written as

$$C^F(R_1, \ldots, R_J, Q_1, \ldots, Q_J) = \sum_{j=1}^{J} \left[c_j R_j + v_j Q_j - g_j \int_0^{I_j+R_j+Q_j} (I_j + R_j + Q_j - D_j) f_j(D_j)\,\mathrm{d}D_j \right.$$
$$\left. + B_j \int_{I_j+R_j+Q_j}^{\infty} (D_j - I_j - R_j - Q_j) f_j(D_j)\,\mathrm{d}D_j \right] - g_0 \left(N - \sum_{j=1}^{J} R_j \right).$$

Noting that

$$\int_{I_j+R_j+Q_j}^{\infty} (D_j - I_j - R_j - Q_j) f_j(D_j)\,\mathrm{d}D_j = E[D_j - I_j - R_j - Q_j]^+$$

we can write the expected cost as

$$C^F(R_1,\dots,R_J,Q_1,\dots,Q_J) = \sum_{j=1}^{J}[(c_j-g_j)R_j+(v_j-g_j)Q_j-g_jI_j+g_j\mu_j + (B_j-g_j)E[D_j-I_j-R_j-Q_j]^+] - g_0\left(N-\sum_{j=1}^{J}R_j\right). \quad (1)$$

Thus we wish to minimize (1) subject to the constraints

$$\sum_{j=1}^{J}R_j \leqslant N,$$

$$R_j \geqslant 0 \quad \forall j,$$

$$Q_j \geqslant 0 \quad \forall j.$$

If we can find an algorithm that satisfies Kuhn–Tucker conditions [25], it is an optimal algorithm. The Lagrangian function is

$$L(R_1,\dots,R_J,Q_1,\dots,Q_J,\lambda,\alpha_1,\dots,\alpha_J,\beta_1,\dots,\beta_J) = \sum_{j=1}^{J}[(c_j-g_j)R_j+(v_j-g_j)Q_j-g_jI_j+g_j\mu_j+(B_j-g_j)E[D_j-I_j-R_j-Q_j]^+] - g_0\left(N-\sum_{j=1}^{J}R_j\right)+\lambda\left[\sum_{j=1}^{J}R_j-N\right]-\sum_{j=1}^{J}\alpha_jR_j-\sum_{j=1}^{J}\beta_jQ_j,$$

where λ is a Lagrange multiplier associated with the constraint on the total number of convertible items. α_j and β_j are Lagrange multipliers associated with the nonnegativity constraints on R_j and Q_j, respectively. The Kuhn–Tucker conditions are as follows:

$$\frac{\partial L}{\partial R_j} = c_j - g_j + g_0 - (B_j-g_j)[1-F_j(I_j+R_j+Q_j)] + \lambda - \alpha_j = 0 \quad \forall j, \quad (2)$$

$$\frac{\partial L}{\partial Q_j} = v_j - g_j - (B_j-g_j)[1-F_j(I_j+R_j+Q_j)] - \beta_j = 0 \quad \forall j, \quad (3)$$

$$\lambda\left[\sum_{j=1}^{J}R_j-N\right]=0, \quad (4)$$

$$\alpha_jR_j=0 \quad \forall j, \quad (5)$$

$$\beta_jQ_j=0 \quad \forall j. \quad (6)$$

Using Eqs. (2) and (3), we can derive the following proposition.

Proposition 1. *The optimal solution must satisfy the following equations*:

$$v_j - c_j - g_0 - \beta_j + \alpha_j = \lambda \quad \forall j, \quad (7)$$

$$F_j(I_j + R_j + Q_j) = \frac{B_j - v_j + \beta_j}{B_j - g_j} \quad \forall j. \tag{8}$$

Before we develop an optimization algorithm for this problem, we first derive the following proposition on the optimal solution.

Proposition 2. *Without loss of generality, we reorder items in decreasing order of* $v_j - c_j$. *Then, the optimal* R_j, Q_j*'s must have the following form:*

$$R_1 > 0, \quad Q_1 = 0, \ldots, R_j > 0, \quad Q_j \geqslant 0, \quad R_{j+1} = 0, \quad Q_{j+1} > 0, \ldots \tag{9}$$

Proof. Let $v_j - c_j > v_{j+1} - c_{j+1}$ for adjacent items j and $j+1$. We use a contradiction method to prove this proposition. Suppose $R_j > 0,\ Q_j > 0$ and $R_{j+1} > 0,\ Q_{j+1} > 0$ are optimal.

Case 1. $R_{j+1} > Q_j$.

Let $R'_j = R_j + Q_j,\ Q'_j = 0, R'_{j+1} = R_{j+1} - Q_j,\ Q'_{j+1} = Q_{j+1} + Q_j$ be another solution. Then, clearly the last two parts of the cost function (1) do not change since $R_j + Q_j = R'_j + Q'_j$ and $R_{j+1} + Q_{j+1} = R'_{j+1} + Q'_{j+1}$. The cost difference of the two solutions is as follows:

$$\begin{aligned} &C^F(R_1, \ldots, R_j, R_{j+1}, \ldots, R_J, Q_1, \ldots, Q_j, Q_{j+1}, \ldots, Q_J) \\ &\quad - C^F(R_1, \ldots, R'_j, R'_{j+1}, \ldots, R_J, Q_1, \ldots, Q'_j, Q'_{j+1}, \ldots, Q_J) \\ &\quad = [(c_j - g_j)R_j + (v_j - g_j)Q_j + (c_{j+1} - g_{j+1})R_{j+1} + (v_{j+1} - g_{j+1})Q_{j+1}] - [(c_j - g_j)(R_j + Q_j) \\ &\qquad + (c_{j+1} - g_{j+1})(R_{j+1} - Q_j) + (v_{j+1} - g_{j+1})(Q_{j+1} + Q_j)] = Q_j[(v_j - c_j) - (v_{j+1} - c_{j+1})] > 0. \end{aligned}$$

This contradicts the optimality of the current solution of $(R_1, \ldots, R_J, Q_1, \ldots, Q_J)$. Thus, the optimal solution must have the form of (9).

Case 2. $R_{j+1} \leqslant Q_j$.

This case can be proven similarly. □

Remark 1. Let S_j and T_j be the optimal unconstrained order-up-to levels based on c_j and v_j, respectively. From the result of the standard newsvendor problem, we can derive the following equations:

$$S_j = F_j^{-1}\left(\frac{B_j - c_j - g_0}{B_j - g_j}\right), \tag{10}$$

$$T_j = F_j^{-1}\left(\frac{B_j - v_j}{B_j - g_j}\right). \tag{11}$$

It is clear that the optimal solution must satisfy the following inequality:

$$T_j \leqslant I_j + R_j + Q_j \leqslant S_j.$$

Moreover, if $Q_j > 0$, then $T_j = I_j + R_j + Q_j$.

From Proposition 2, we can derive the following Corollary which will be the basis for the optimal algorithm.

Corollary 1. *At most one item can have its* $\alpha_j = \beta_j = 0$. *In addition, for any optimal solution,* $\alpha_j \beta_j = 0$ *for all* j.

Using the above Corollary, we can derive the following algorithm to find an optimal solution. The algorithm is basically a line search on λ.

Step 1 (Optimal line search algorithm). We check whether the unconstrained order-up-to levels, S_js, satisfy the constraint on the available number of units of the convertible item or not as follows:

$$\sum_{j=1}^{J}(S_j - I_j) \leqslant N.$$

If they satisfy the above constraint, they are indeed optimal. Otherwise, go to (Step 2).

Step 2. Start with an arbitrary $\lambda > 0$.

Step 3. Compute α_j and β_j sequentially using (7) and $\alpha_j \beta_j = 0$.

Step 4. Using α_j and β_j computed in (Step 3), we know whether $Q_j = 0$ or $Q_j > 0$ (i.e. $R_j > 0$ or $R_j = 0$). There is a possibility that both $R_j > 0$ and $Q_j > 0$ for one item (if λ happens to equal $v_j - c_j - g_0$).

Step 5. Using (8), compute the R_js and Q_js. In the case of both $\alpha_j = \beta_j = 0$, we assign $R_j = \max\{N - \sum_{k<j} R_k, 0\}$. If $R_j = N - \sum_{k<j} R_k$ and the corresponding Q_j becomes negative, then set $R_j = 0$.

Step 6. If $\sum_j R_j < N$, then decrease λ and go to (Step 3).

If $\sum_j R_j > N$, then increase λ and go to (Step 3).

If $\sum_j R_j = N$, we have found an optimal solution.

Remark 2. We can also solve this problem using dynamic programming.

Example. We assume independent normal distributions and that there are 150 units of the convertible item available. The detailed data for this example are given in Table 1. We set $g_0 = \$5$. The unconstrained optimal order-up-to levels have been computed in Table 2. Note that the Ts and Ss will not be integers in general. We have rounded them to the nearest integers.

Let $(R_1^N, \ldots, R_J^N, Q_1^N, \ldots, Q_J^N)$ be the optimal solution for the normal distribution. The results for the example are $(R_1^N, Q_1^N) = (70, 0)$, $(R_2^N, Q_2^N) = (71, 0)$, $(R_3^N, Q_3^N) = (9, 71)$, $(R_4^N, Q_4^N) = (0, 165)$ with a total cost of \$76,076.21. Note that $\lambda^* = 15$ is equal to $v_3 - c_3 - g_0$, which has produced positive values of both R_3 and Q_3. Suppose we change both I_1 and I_3 to $I_1 = 80$, $I_3 = 41$. Then the solution becomes

Table 1
Data for the example ($N = 150$)

Item	v_j (\$)	c_j (\$)	g_j (\$)	B_j (\$)	μ_j	σ_j	I_j
1	300	150	125	400	80	20	30
2	400	351	250	503	90	25	20
3	300	280	151	320	120	17	20
4	50	40	20	70	230	60	50

Table 2
Unconstrained optimal order-up-to levels

Item	T_j	S_j
1	73	105
2	84	95
3	100	106
4	215	230

$(R_1^N, Q_1^N) = (20, 0)$, $(R_2^N, Q_2^N) = (71, 0)$, $(R_3^N, Q_3^N) = (59, 0)$, $(R_4^N, Q_4^N) = (0, 165)$ with total cost of \$61,276.21. The optimal Lagrange multiplier, λ^*, is 14.85 for this case.

2.4. Illustration of the value of convertible units available

By repeating the above algorithm for different integer values of N, we can develop a table or graphical representation of the value of having different numbers of convertible units available. This is illustrated in Fig. 1 and Table 1 for N from 0 to 300. If there are options of acquiring different numbers of convertible units, with associated costs, then one can use the results of Fig. 1 or Table 1 to choose among the options. If we compute the incremental savings in Table 1, we can notice that the cost savings are marginally decreasing (Table 3).

3. Distribution free model

In practice, the distributional information about the demand is often limited. Sometimes all that is available are estimates of the mean and variance. There is a tendency to use the normal distribution under these conditions. However, the normal distribution does not offer the best protection against the occurrences of other distributions with the same mean and variance. Scarf [21] addressed a newsvendor problem where only the mean μ and the variance σ^2 of the demand are known without any further assumptions about the form of the distribution of the demand. Taking a conservative approach, he modeled the problem as that of finding the order quantity that maximizes the expected profit against the worst possible distribution of the demand with mean μ and variance σ^2. The approach is called the minimax distribution free

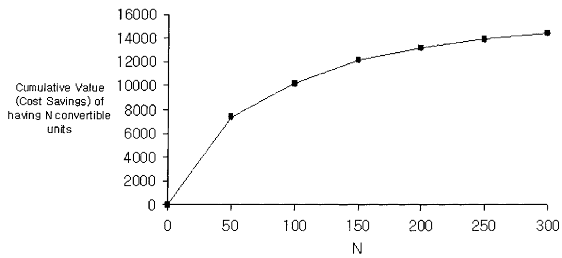

Fig. 1. Cumulative value (cost savings) of having different numbers of convertible units available.

Table 3
Cumulative value (cost savings) of having different numbers of convertible units available

N	Optimal cost (\$)	Cost savings of having N units of convertible items (\$)
0	88,247.51	–
50	80,876.97	7370.54
100	78,048.49	10,199.02
150	76,076.21	12,171.30
200	75,076.21	13,171.30
250	74,316.55	13,930.96
300	73,816.55	14,430.96

approach. Recently, there have been many related papers. Gallego and Moon [8] presented a very compact proof of the optimality of Scarf's ordering rules for the newsvendor problem and extended the analysis to several cases including a fixed ordering cost, multiple products, random yield, and the possibility of recourse. Moon and Gallego [18] applied the approach to several inventory models including both continuous and periodic review models. Moon and Choi [15] extended the model of Gallego and Moon [8] to the case that allows customers to balk when inventory level is low. Shore [23] derived explicit approximate solutions to the standard newsboy problem, to some (Q, r) models, and to a periodic review model in which the first three or four moments of the demand are known. Moon and Choi [16] applied the approach to the two-echelon stochastic production/inventory models in which assemble-to-order (ATO), assemble-to-make (ATM), and composite policies can be adopted. Hariga [11] extended the work of Moon and Choi [16] to the multi-echelon case. Gallego et al. [9] considered stochastic finite-horizon inventory models with discrete demand distributions that are incompletely specified by selected moments, percentiles, or a combination of moments and percentiles. Hariga and Ben Daya [10] and Moon and Choi [17] independently solved the continuous review inventory problem in which lead time can be reduced by investment. Their models generalized the distribution free model of Ouyang and Wu [20] by simultaneously optimizing both the order quantity and reorder point.

Thus, we now consider the distribution free approach, i.e. we make no assumption on the distribution F of D_j other than saying that it belongs to the class $\mathscr{F}$ of cumulative distribution functions with mean μ_j and variance σ_j^2. Since the distribution F of D is unknown we want to minimize the total expected cost against the worst possible distribution in $\mathscr{F}$. The distribution free approach for this model involves finding the most unfavorable distribution in $\mathscr{F}$ for each $(R_1, \ldots, R_J, Q_1, \ldots, Q_J)$. Our problem is to solve:

$$\min_{R_1,\ldots,R_J,Q_1,\ldots,Q_J} \max_{F\in\mathscr{F}} C^F(R_1, \ldots, R_J, Q_1, \ldots, Q_J)$$

$$\text{subject to} \quad \sum_{j=1}^{J} R_j \leqslant N, \qquad R_j \geqslant 0 \quad \forall j, \qquad Q_j \geqslant 0 \quad \forall j.$$

To this end, we need to use the following proposition as in Gallego and Moon [8].

Proposition 3. *For any* $F \in \mathscr{F}$

$$E[D_j - I_j - R_j - Q_j]^+ \leqslant \frac{1}{2}\left\{\sqrt{\sigma_j^2 + (I_j + R_j + Q_j - \mu_j)^2} - (I_j + R_j + Q_j - \mu_j)\right\}.$$

Moreover, the upper bound is tight. In other words, we can always find a distribution in which the above bound is satisfied with equality for every R_j *and* Q_j.

Using the above proposition, our problem becomes

$$\min_{R_1,\ldots,R_J,Q_1,\ldots,Q_J} \sum_{j=1}^{J}\Bigg[(c_j - g_j)R_j + (v_j - g_j)Q_j - g_jI_j + g_j\mu_j + \frac{B_j - g_j}{2}\left\{\sqrt{\sigma_j^2 + (I_j + R_j + Q_j - \mu_j)^2} - (I_j + R_j + Q_j - \mu_j)\right\}\Bigg] - g_0\left(N - \sum_{j=1}^{J} R_j\right)$$

$$\text{subject to} \quad \sum_{j=1}^{J} R_j \leqslant N, \quad R_j \geqslant 0 \quad \forall j, \quad Q_j \geqslant 0 \quad \forall j.$$

We use the same approach as in the previous section to find an optimal solution. The Lagrangian function is

$$\begin{aligned} L(R_1,\ldots,R_J,Q_1,\ldots,Q_J,\lambda,\alpha_1,\ldots,\alpha_J,\beta_1,\ldots,\beta_J) \\ = \sum_{j=1}^{J}\Bigg[(c_j-g_j)R_j+(v_j-g_j)Q_j-g_jI_j+g_j\mu_j \\ +\frac{B_j-g_j}{2}\left\{\sqrt{\sigma_j^2+(I_j+R_j+Q_j-\mu_j)^2}-(I_j+R_j+Q_j-\mu_j)\right\}\Bigg] \\ -g_0\left(N-\sum_{j=1}^{n}R_j\right)+\lambda\left[\sum_{j=1}^{J}R_j-N\right]-\sum_{j=1}^{J}\alpha_jR_j-\sum_{j=1}^{J}\beta_jQ_j. \end{aligned}$$

Kuhn–Tucker conditions are as follows:

$$\frac{\partial L}{\partial R_j}=c_j-g_j+g_0-\frac{B_j-g_j}{2}\left\{1-\frac{I_j+R_j+Q_j-\mu_j}{\sqrt{\sigma_j^2+(I_j+R_j+Q_j-\mu_j)^2}}\right\}+\lambda-\alpha_j=0 \quad \forall j, \tag{12}$$

$$\frac{\partial L}{\partial Q_j}=v_j-g_j-\frac{B_j-g_j}{2}\left\{1-\frac{I_j+R_j+Q_j-\mu_j}{\sqrt{\sigma_j^2+(I_j+R_j+Q_j-\mu_j)^2}}\right\}-\beta_j=0 \quad \forall j, \tag{13}$$

$$\lambda\left[\sum_{j=1}^{J}R_j-N\right]=0, \tag{14}$$

$$\alpha_jR_j=0 \quad \forall j, \tag{15}$$

$$\beta_jQ_j=0 \quad \forall j. \tag{16}$$

Using Eqs. (12) and (13), we can derive the following proposition which is similar to Proposition 1 in the previous section.

Proposition 4. *The optimal solution must satisfy the following equations*:

$$v_j-c_j-g_0-\beta_j+\alpha_j=\lambda \quad \forall j, \tag{17}$$

$$\frac{I_j+R_j+Q_j-\mu_j}{\sqrt{\sigma_j^2+(I_j+R_j+Q_j-\mu_j)^2}}=\frac{B_j-2v_j+g_j+2\beta_j}{B_j-g_j} \quad \forall j. \tag{18}$$

We can use the same algorithm as in the previous section to find an optimal solution except that Eq. (8) is replaced by Eq. (18).

Let the optimal solutions for distribution F and for the worst distribution be $(R_1^F, \ldots, R_J^F, Q_1^F, \ldots, Q_J^F)$ and $(R_1^W, \ldots, R_J^W, Q_1^W, \ldots, Q_J^W)$, respectively. We are interested in the *Expected Value of Additional Information* (EVAI) [8], i.e.

$$C^F(R_1^W, \ldots, R_J^W, Q_1^W, \ldots, Q_J^W) - C^F(R_1^F, \ldots, R_J^F, Q_1^F, \ldots, Q_J^F). \tag{19}$$

This is the largest amount that we would be willing to pay for the knowledge of the specific distribution in the absence of knowledge of the specific distribution. If this quantity is small, the use of the distribution free solution can be justified.

Example. We solve the same example as in the previous section to illustrate the distribution free approach. We compare the performance of the normal solution of $(R_1^N, \ldots, R_J^N, Q_1^N, \ldots, Q_J^N)$ with the distribution free solution $(R_1^W, \ldots, R_J^W, Q_1^W, \ldots, Q_J^W)$.

The results for the base case (of Table 1) are $(R_1^W, Q_1^W) = (68, 0)$, $(R_2^W, Q_2^W) = (71, 0)$, $(R_3^W, Q_3^W) = (11, 69)$, $(R_4^W, Q_4^W) = (0, 168)$ with total cost of \$76,082.00, and the EVAI is

$$C^N(R_1^W, \ldots, R_4^W, Q_1^W, \ldots, Q_4^W) - C^N(R_1^N, \ldots, R_4^N, Q_1^N, \ldots, Q_4^N) = \$76,082.00 - \$76,076.21 = \$5.79$$

Suppose we again change both I_1 and I_3 to $I_1 = 80$, $I_3 = 41$. Then the solution becomes $(R_1^W, Q_1^W) = (19, 0)$, $(R_2^W, Q_2^W) = (71, 0)$, $(R_3^W, Q_3^W) = (60, 0)$, $(R_4^W, Q_4^W) = (0, 168)$ with total cost of \$61,279.43, and the EVAI is

$$C^N(R_1^W, \ldots, R_4^W, Q_1^W, \ldots, Q_4^W) - C^N(R_1^N, \ldots, R_4^N, Q_1^N, \ldots, Q_4^N) = \$61,279.43 - \$61,276.21 = \$3.22.$$

From this example, we can conjecture that the distribution free approach is very robust. We will confirm this in the next section.

4. Computational results

In order to further investigate the robustness of the distribution free approach, 25 test problem instances were generated randomly from uniform distributions on the given intervals as shown below:

$J \sim DU(10, 20)$ where DU denotes a discrete uniform distribution,

$$v \sim U(300, 500), \quad c \sim U(0.5, 0.9) \times v, \quad g \sim U(0.5, 0.7) \times c, \quad B \sim U(1.2, 1.5) \times v,$$

$$\mu \sim U(100, 300), \quad \sigma \sim U(0.1, 0.3) \times \mu, \quad I \sim U(0.1, 0.5) \times \mu \quad N \sim U(0.1, 0.5) \times \sum_{j=1}^{n} \mu_j.$$

We compute the following ratio to test the robustness of the distribution free solution:

$$\frac{C^F(R_1^W, \ldots, R_J^W, Q_1^W, \ldots, Q_J^W)}{C^F(R_1^F, \ldots, R_J^F, Q_1^F, \ldots, Q_J^F)}.$$

We report the minimum, maximum and mean values of this ratio in Table 4. The ratios are very close to 1 which indicates that use of the distribution free solution in the absence of the specific form of the distribution function produces negligible penalties when the true distributions are normal.

Table 4
Results of computational tests

Minimum ratio	Maximum ratio	Mean ratio
1.00005	1.00021	1.00012

5. Conclusions

In this paper, we have dealt with a multiple end item, single period problem where, besides having the usual option of purchasing units of the end items, one has a number of units of a convertible item available that can be converted into units of the end items. Efficient algorithms have been developed for the cases of: (i) known distributions of end item demands and (ii) a distribution free approach where only the mean and standard deviation of demand for each item are assumed known. As mentioned in the Introduction, a single period formulation is clearly an approximation of the practical situations that motivated our interest in analysis of decisions involving the use of convertible items. An obvious, but difficult, extension would be to the case of multiple periods of demand where unused convertible and/or end item units could be carried forward in stock. Another extension would be to the case where the conversion and purchase lead times are not the same, in particular conversion might be quicker and one could wait to partially observe the demand before carrying out some of the conversion activity.

There are at least two possible extensions to this work. The first is where there is more than one class of convertible items. The second is where there is ongoing demand for more than one period. We hope to report at a later date on useful results in these two directions.

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