

Minimizing the expected total value of shortages for a population of items subject to practical restrictions on the reorder points

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Abstract

We address a problem of setting reorder points (expressed as time supplies) of a population of items, subject to a restricted set of possible time supplies as well as a budget on the total amount of safety stocks, both important practical constraints. We provide a branch-and-bound algorithm for obtaining the optimal solution. In addition, a simple and efficient heuristic algorithm has been developed. Computational experiments show that the performance of the heuristic is excellent based on a set of realistic examples. However, the typical set of possible time supplies may significantly degrade performance compared with the situation where a continuum of choices are possible. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Managers are concerned with the effective use of limited resources. An example is the allocation of a specific budget among the safety stocks of a population of items, each of which is controlled by a continuous review, order point and order quantity system. One possible criterion is the minimization of an aggregate measure of disservice, such as the expected total value of units short per year (ETVSPY). The case where there is no restriction,

other than the budget constraint, on the choice of safety stocks (or reorder points) has been treated in the literature [1]. In fact, by varying the levels of the available budget, one can trace out a whole exchange curve which shows the best that one can do on the service measure as the budget is varied (see, for example, [2] or [3]). However, these results hinge upon continuous possible values of the decision variables (the reorder points or, equivalently, the safety stocks); but, in practice, managers often prefer to restrict the decision variables to a set of easily understood and implementable discrete values, e.g. the reorder point, expressed as a time supply, which is restricted to one of the following values: 1 week, 2 weeks, 1 month, 2 months, 3 months, 6 months or 1 year.

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This paper addresses how to deal with the above-constrained problem in a pragmatic fashion. Specifically, we treat the case where there is a set of time supplies, one of which must be used for each of a population of items. There is a specified upper limit on the total value of safety stock to be used and we wish to choose the time supply reorder points of the population of items, subject to a discrete set of options and the aggregate constraint, so as to minimize the expected total values of the units short per year.

In the next section, we introduce the notation and mathematically formulate the problem, including obtaining a useful lower bound. This is followed by the specification of an optimal solution procedure using a branch-and-bound algorithm. The associated computational effort required increases substantially with the number of items. Perhaps more important, branch and bound is a very difficult concept to explain to practitioners. Therefore, we subsequently present a heuristic approach that overcomes both of these drawbacks. Moreover, we present results of computational experiments that show that very little degradation in the objective function value results from using the heuristic instead of the optimal solution. However, the performance of the optimal solution is markedly affected by the discrete choice of the time supplies, particularly if the set is fairly sparse as would likely be the case in practice.

2. Problem formulation and a lower bound

The notation to be used is as follows:

- n number of items
- D_i the demand rate of item i , in units/unit time, $i = 1, \dots, n$
- v_i the unit variable cost of item i , in monetary units/unit, $i = 1, \dots, n$
- \hat{x}_i the mean lead time demand of item i , $i = 1, \dots, n$
- σ_i the standard deviation of lead time demand of item i , $i = 1, \dots, n$
- Q_i the predetermined order quantity of item i , $i = 1, \dots, n$
- Y total budget for safety stocks

- t_i the reorder point of item i (expressed as a time supply), $i = 1, \dots, n$
- T the discrete set of possible time supplies
- m number of possible discrete reorder time supplies, and
- T_j the j th possible reorder time supply, $j = 1, \dots, m$

There are many different methods of selecting safety stocks (or reorder points) in control systems under probabilistic demand. In this paper, we use the criterion of minimization of expected total value short per year subject to a specified total safety stock. The assumptions are exactly the same as for the classical multi-item (s, Q) inventory model with a budget constraint on the total safety stock (see [3]) except that we add the important practical consideration that the time supplies are limited to a discrete set T . Without loss of generality, we assume that the items are numbered such that $D_1 v_1 \geq D_2 v_2 \geq \dots \geq D_n v_n$. We assume that lead time demand follows a normal distribution. Define $G_u(k)$ as

$$G_u(k) = \int_k^\infty (u - k) \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du,$$

which measures the expected number of units of demand in excess of supply in each replenishment cycle.

Then our problem can be represented as follows:

(ETVSPY)

$$\text{Min ETVSPY} = \sum_{i=1}^n \frac{D_i}{Q_i} \sigma_i v_i G_u\left(\frac{D_i t_i - \hat{x}_i}{\sigma_i}\right)$$

s.t.

$$\sum_{i=1}^n D_i v_i t_i \leq Y', \tag{1}$$

$$t_i \in \{T_1, T_2, \dots, T_m\}, \quad \forall i, \tag{2}$$

where

$$Y' = Y + \sum_{i=1}^n \hat{x}_i v_i.$$

We derive a lower bound on (ETVSPY) for two reasons. First, the lower bound is the optimal value of the objective function when the t_i 's are not restricted to a discrete set of values. Thus, it will provide an indication of the degradation (increase

in the expected total value short per year) caused by the introduction of the pragmatic constraint of restricting the t_i 's to the discrete set T . Second the lower bound solution will be used as a starting point in our heuristic approach.

If we relax constraint (2), we obtain the following relaxed version of (ETVSPY):

$$(LB) \quad \text{Min ETVSPY} = \sum_{i=1}^n \frac{D_i}{Q_i} \sigma_i v_i G_u \left(\frac{D_i t_i - \hat{x}_i}{\sigma_i} \right)$$

s.t.

$$\sum_{i=1}^n D_i v_i t_i \leq Y'. \tag{3}$$

Note that constraint (3) holds as a strict equality. Observe that the objective function is a convex decreasing function of t_i , $\forall i$. Note that $\lambda > 0$ is an optimal Lagrange multiplier.

If we can find a Karush–Kuhn–Tucker (KKT) solution, it will be a global minimum due to the fact that the objective function is convex and the set of feasible solutions is a convex set and is non-empty if $Y' > 0$. The Lagrangian function is as follows:

$$\begin{aligned} L(t_1, \dots, t_n, \lambda) &= \sum_{i=1}^n \frac{D_i}{Q_i} \sigma_i v_i G_u \left(\frac{D_i t_i - \hat{x}_i}{\sigma_i} \right) \\ &\quad + \lambda \left(\sum_{i=1}^n D_i v_i t_i - Y' \right). \end{aligned}$$

The Karush–Kuhn–Tucker conditions require (see also [3])

$$\begin{aligned} \frac{\partial L}{\partial t_i} &= \left(\frac{D_i \sigma_i v_i}{Q_i} \right) \left\{ - \frac{D_i}{\sigma_i} p_{u \geq} \left(\frac{D_i t_i - \hat{x}_i}{\sigma_i} \right) \right\} \\ &\quad + \lambda D_i v_i = 0, \quad \forall i, \end{aligned}$$

where

$$p_{u \geq}(k) = \int_k^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du$$

is the right-hand tail of the unit normal distribution and represents the probability of a stockout during a replenishment lead time. Simplifying, we obtain

$$\lambda = \frac{D_i}{Q_i} p_{u \geq} \left(\frac{D_i t_i - \hat{x}_i}{\sigma_i} \right) \tag{4}$$

and

$$\sum_{i=1}^n D_i v_i t_i = Y'. \tag{5}$$

Note that Eq. (4) implies that the expected number of cycles per year during which stockouts occur is the same for all items and that this value equals the value of the Lagrangian multiplier. We can use the following line search algorithm to find a solution that satisfies (4) and (5).

Step 1: Start from an arbitrary $\lambda > 0$.

Step 2: Using λ and (4), find t_i , $\forall i$.

Step 3: If $\sum_{i=1}^n D_i v_i t_i = Y'$, stop. We find an associated optimal solution for (LB), say t_i^0 's.

If $\sum_{i=1}^n D_i v_i t_i < Y'$, decrease λ and go to Step 2.

If $\sum_{i=1}^n D_i v_i t_i > Y'$, increase λ and go to Step 2.

We now illustrate the above algorithm using an example found in [3], which is reproduced in Table 1. The three items are produced and stocked by a company. It is not uncommon for organizations to use the following type of rule for setting reorder points for a broad range of items: reorder when the inventory position has dropped to a specific time supply. We assume that the current reorder points of the company are each based on a two-month time supply (that is, $D/6$).

Assuming normally distributed lead time demand, the safety stock and ETVSPY using the current policy are as listed under the title of 'equal time supply' in Table 2. If we use the algorithm to find the lower bound solution presented above,

Table 1
Data for the 3 item example

Item	D (units/yr)	v (\$/unit)	\hat{x} (units)	σ (units)	Q (units)	s (units) ^a
PSP-001	6000	20	750	125	6000	1000
PSP-002	3000	10	375	187.5	1000	500
PSP-003	2400	12	300	62.5	1200	400

^as: reorder point.

Table 2
Results for the 3 item example

Item no.	Equal time supply		Optimal	
	SS ^a	ETVSPY	SS ^a	ETVSPY
1	\$5000	\$21	\$2972	\$143.4
2	\$1250	\$845	\$3302	\$88.5
3	\$1200	\$35	\$1175	\$37.7
Total	\$7450	\$901/yr	\$7450	\$269.6/yr

^aSS: safety stock, ETVSPY: expected total value short per year.

which is equivalent to the optimal solution under continuous possible time supplies, we can achieve 70% savings compared to the strategy based on equal probabilities of stockout per cycle.

3. Branch-and-bound algorithm for optimal solution

Because of the discrete nature of the T_j 's we develop a branch-and-bound scheme to solve (ETVSPY). Since we use the heuristic (which has been presented in Section 4) as an upper bound and its performance is quite good (which will be demonstrated later), we believe that most branches will be fathomed in early stages, especially branches with small t values. In addition, the budget constraint also plays a role in eliminating many branches in early stages, especially branches with large t values. Nevertheless, we conjecture that (ETVSPY) is an NP-hard problem [4], hence there is a need, from a computational standpoint, for a heuristic solution procedure for large instances of the problem. The problem is a knapsack like problem and hence inherits its computational complexity from this problem.

The branch-and-bound algorithm is outlined below.

Step 1: Start from item 1 (the item with the largest Dv value).

(i) *Branching:* Produce m branches in which each branch corresponds to one value from T .

(ii) *Bounding:* We use the value of the heuristic as an initial upper bound (VAL_U). If we find a better solution during the branch-and-bound algorithm, it will be a new upper bound.

(iii) *Fathoming:* A branch will be fathomed if either of the following condition is satisfied.

(a) If the t_1 value and the smallest assignment of t_2 to t_n spend more than the budget, the branch will be fathomed. That is,

If

$$D_1 v_1 t_1 + \sum_{i=2}^n D_i v_i T_1 > Y', \quad \text{the branch is fathomed.}$$

(b) If the t_1 value and the largest assignment of t_2 to t_n give an objective value larger than the current upper bound, the branch is fathomed. That is,

If

$$\frac{D_1}{Q_1} \sigma_1 v_1 G_u \left(\frac{D_1 t_1 - \hat{x}_1}{\sigma_1} \right) + \sum_{i=2}^n \frac{D_i}{Q_i} \sigma_i v_i G_u \left(\frac{D_i T_m - \hat{x}_i}{\sigma_i} \right) > VAL_U,$$

the branch will be fathomed.

Step 2:

(i) *Branching:*

Produce m branches for each of the branches not fathomed in Step 1.

(ii) *Bounding:* same as before.

(iii) *Fathoming:*

(a) If the t_1 and t_2 values and the smallest assignment of t_3 to t_n spend more than the budget, the branch is fathomed. That is,

If

$$\sum_{i=1}^2 D_i v_i t_i + \sum_{i=3}^n D_i v_i T_1 > Y',$$

the branch will be fathomed.

(b) If the t_1 and t_2 values and the largest assignment of t_3 to t_n give an objective value larger than the current upper bound, the branch is

fathomed. That is,

If

$$\sum_{i=1}^2 \frac{D_i}{Q_i} \sigma_i v_i G_u \left(\frac{D_i t_i - \hat{x}_i}{\sigma_i} \right) + \sum_{i=3}^n \frac{D_i}{Q_i} \sigma_i v_i G_u \left(\frac{D_i T_m - \hat{x}_i}{\sigma_i} \right) > \text{VAL}_U,$$

the branch will be fathomed.

Repeat this procedure until all the branches are either searched or fathomed.

Note: We could have developed a dynamic programming formulation for this problem similar to that of the knapsack problem. However, the size of the problem would become enormous due to the noninteger nature of the Dvt values.

The branch-and-bound algorithm has been coded using GAUSS [5], and it has been run on an IBM Pentium II PC with 266 MHz clock speed. A depth-first-search method has been used to avoid a memory problem.

4. Heuristic algorithm

We develop a heuristic algorithm for (ETVSPY) since, as mentioned earlier, it would take much time to solve the branch-and-bound algorithm if n becomes quite large. Moreover, the heuristic solution is much easier for a practitioner to understand than is a branch-and-bound algorithm. Note that we can obtain an upper bound directly from the lower bound solution of the t_i 's. That is, if we round the t_i^0 's down to the nearest T_j 's, we obtain an upper bound (a feasible solution) immediately. The heuristic algorithm is based on starting with the lower bound (infeasible) solution and then using a marginal (or greedy) allocation algorithm [6] to change the solution. At each step of the algorithm, we find the item that least increases the objective value per unit decrease of the stock investment. This item's reorder point is then reduced to the next lower value in T . This is continued until a feasible solution is obtained. Any remaining capacity is filled in a reverse greedy fashion.

Step 1: Solve the lower bound problem. Let t_i^0 be the lower bound solution for t_i .

Step 2: Suppose $T_{r_i} < t_i^0 < T_{r_i+1}$ (if $t_i^0 = T_{r_i}$ or T_{r_i+1} , we just leave t_i^0 as it is). Round up all t_i^0 's obtained in solving (LB) to the nearest T_j (this solution will always be infeasible unless $t_i^0 \in T = \{T_1, \dots, T_m\} \forall i$ since the t_i^0 's satisfy constraint (1) as an equality). Let the current value of t_i be T_{l_i} .

Step 3: Choose the item i such that

$$\left\{ \frac{\sigma_i v_i (D_i / Q_i) [G_u((D_i T_{l_i-1} - \hat{x}_i) / \sigma_i) - G_u((D_i T_{l_i} - \hat{x}_i) / \sigma_i)]}{D_i v_i (T_{l_i} - T_{l_i-1})} \right\}$$

is minimized.

(6)

Then decrease t_i from T_{l_i} to T_{l_i-1} , and update t_i .

Step 4: If $\sum_{i=1}^n D_i v_i t_i = Y'$, stop. We have found a heuristic solution.

If $\sum_{i=1}^n D_i v_i t_i < Y'$, go to Step 5.

If $\sum_{i=1}^n D_i v_i t_i > Y'$, go to Step 3.

Step 5: Let the current value of t_i be $T_{l_i} \forall i$ after the above steps. Let $N = \{1, 2, \dots, n\}$. Choose the item $i \in N$ for which

$$\left\{ \frac{\sigma_i v_i (D_i / Q_i) [G_u((D_i T_{l_i} - \hat{x}_i) / \sigma_i) - G_u((D_i T_{l_i+1} - \hat{x}_i) / \sigma_i)]}{D_i v_i (T_{l_i+1} - T_{l_i})} \right\}$$

is maximized.

(7)

If $D_i v_i (T_{l_i+1} - T_{l_i}) \leq Y' - \sum_{i=1}^n D_i v_i t_i$, increase t_i from T_{l_i} to T_{l_i+1} and update t_i . If not, set $N \leftarrow N - \{i\}$ and repeat this step as long as $N \neq \phi$.

Numerical example. We solved a randomly generated 24 item problem (The method of generation will be explained in Section 5). The data for this problem are in Table 3.

For illustrative purposes we set the budget for safety stock at 1451, which implies a Y' value of 19562. The reorder point time supplies are restricted to $T = \{1 \text{ week}, 2 \text{ weeks}, 3 \text{ weeks}, 1 \text{ month}, 2 \text{ months}, 3 \text{ months}, 4 \text{ months}, 5 \text{ months}, 6 \text{ months}\}$. The optimal solution for this example, using the branch-and-bound algorithm, is as follows (where we have shown all time supplies as fractions of a year):

$$t_1^* = \frac{1}{12}, \quad t_2^* = \frac{3}{12}, \quad t_3^* = \frac{2}{12}, \quad t_4^* = \frac{3}{12}, \quad t_5^* = \frac{3}{12},$$

Table 3
Data for the 24 item example and lower bound, optimal, and heuristic solutions^a

Item	D_v	Q_v	\hat{x}_v	σ_v	t^0	t^*	t^H
1	28717	2076	2187.51	48.18	0.07888	$\frac{1}{12}$	$\frac{1}{12}$
2	21732	1949	5396.10	142.25	0.25815	$\frac{3}{12}$	$\frac{3}{12}$
3	14671	1571	1761.86	357.47	0.15447	$\frac{2}{12}$	$\frac{2}{12}$
4	7630	932	1501.02	236.29	0.23818	$\frac{3}{12}$	$\frac{3}{12}$
5	7269	1615	1617.73	80.23	0.23334	$\frac{3}{12}$	$\frac{3}{12}$
6	5021	1202	877.38	24.95	0.17935	$\frac{2}{12}$	$\frac{2}{12}$
7	4871	1176	307.35	37.14	0.07013	$\frac{3}{52}$	$\frac{3}{52}$
8	3456	969	756.58	52.28	0.23127	$\frac{3}{12}$	$\frac{3}{12}$
9	3295	703	709.48	54.92	0.23207	$\frac{3}{12}$	$\frac{3}{12}$
10	3134	869	482.90	30.71	0.16216	$\frac{2}{12}$	$\frac{2}{12}$
11	3092	581	134.69	9.72	0.04697	$\frac{3}{52}$	$\frac{3}{52}$
12	3008	1117	574.34	22.18	0.19538	$\frac{2}{12}$	$\frac{2}{12}$
13	2305	934	451.71	49.86	0.20740	$\frac{2}{12}$	$\frac{2}{12}$
14	2230	710	362.46	54.43	0.18011	$\frac{2}{12}$	$\frac{2}{12}$
15	1558	665	39.48	8.07	0.02784	$\frac{1}{52}$	$\frac{1}{52}$
16	1485	466	276.15	14.78	0.19323	$\frac{2}{12}$	$\frac{2}{12}$
17	1046	404	255.71	10.76	0.25019	$\frac{3}{12}$	$\frac{3}{12}$
18	877	529	112.50	28.61	0.13267	$\frac{1}{12}$	$\frac{1}{12}$
19	661	483	148.90	13.01	0.22333	$\frac{2}{12}$	$\frac{2}{12}$
20	579	409	30.79	9.00	0.05233	$\frac{1}{52}$	$\frac{2}{52}$
21	532	463	83.21	15.68	0.14569	$\frac{1}{12}$	$\frac{3}{52}$
22	198	278	8.52	3.24	0.00001	$\frac{1}{52}$	$\frac{1}{52}$
23	198	271	17.78	7.39	0.00001	$\frac{1}{52}$	$\frac{1}{52}$
24	194	134	17.10	4.89	0.08745	$\frac{1}{12}$	$\frac{1}{12}$
ETVSPY					\$511.78	\$1581.77	\$1582.74

^a t^0 : lower bound solution, t^* : optimal solution, t^H : heuristic solution.

$$t_6^* = \frac{2}{12}, t_7^* = \frac{3}{52}, t_8^* = \frac{3}{12}, t_9^* = \frac{3}{12}, t_{10}^* = \frac{2}{12},$$

$$t_{11}^* = \frac{3}{52}, t_{12}^* = \frac{2}{12}, t_{13}^* = \frac{2}{12}, t_{14}^* = \frac{2}{12}, t_{15}^* = \frac{1}{52},$$

$$t_{16}^* = \frac{2}{12}, t_{17}^* = \frac{3}{12}, t_{18}^* = \frac{1}{12}, t_{19}^* = \frac{2}{12}, t_{20}^* = \frac{1}{52},$$

$$t_{21}^* = \frac{1}{12}, t_{22}^* = \frac{1}{52}, t_{23}^* = \frac{1}{52}, t_{24}^* = \frac{1}{12}.$$

The optimal objective value is 1581.77.

We also solved the same 24 item example, but this time using the heuristic algorithm.

Step 1: First, we compute the lower bound solution using the line search algorithm in Section 2.

The lower bound solution is as follows:

$$t_1^0 = 0.07888, t_2^0 = 0.25815, t_3^0 = 0.15447,$$

$$t_4^0 = 0.23818, t_5^0 = 0.23334, t_6^0 = 0.17935,$$

$$t_7^0 = 0.07013, t_8^0 = 0.23127, t_9^0 = 0.23207,$$

$$t_{10}^0 = 0.16216, t_{11}^0 = 0.04697, t_{12}^0 = 0.19538,$$

$$t_{13}^0 = 0.20740, t_{14}^0 = 0.18011, t_{15}^0 = 0.02784,$$

$$t_{16}^0 = 0.19323, t_{17}^0 = 0.25019, t_{18}^0 = 0.13267,$$

$$t_{19}^0 = 0.22333, t_{20}^0 = 0.05233, t_{21}^0 = 0.14569,$$

$$t_{22}^0 = 0.00001, t_{23}^0 = 0.00001, t_{24}^0 = 0.08745.$$

The value of the lower bound is 511.78.

Step 2: Round up the t_i^0 's to the nearest T_j 's, which provides an initial solution for the heuristic. This solution is clearly infeasible. Note that the

objective value of the infeasible solution is 104.17.

$$t_1 = \frac{1}{12}, t_2 = \frac{4}{12}, t_3 = \frac{2}{12}, t_4 = \frac{3}{12}, t_5 = \frac{3}{12},$$

$$t_6 = \frac{3}{12}, t_7 = \frac{1}{12}, t_8 = \frac{3}{12}, t_9 = \frac{3}{12}, t_{10} = \frac{2}{12},$$

$$t_{11} = \frac{3}{52}, t_{12} = \frac{3}{12}, t_{13} = \frac{3}{12}, t_{14} = \frac{3}{12}, t_{15} = \frac{2}{52},$$

$$t_{16} = \frac{3}{12}, t_{17} = \frac{4}{12}, t_{18} = \frac{2}{12}, t_{19} = \frac{3}{12}, t_{20} = \frac{3}{52},$$

$$t_{21} = \frac{2}{12}, t_{22} = \frac{1}{52}, t_{23} = \frac{1}{52}, t_{24} = \frac{2}{12}.$$

Steps 3 and 4: We compute the priority ratios of the reduction of the t values dynamically using Eq. (6). The priority order of the reduction of the t values is as follows:

- 17, 24, 2, 14, 6, 16, 12, 15, 18, 20, 13, 21, 19,
7, 24, 21.

It means that we first reduce t_{17} from $\frac{4}{12}$ to $\frac{3}{12}$, and check whether $\sum_{i=1}^n D_i v_i t_i \leq Y'$. We iterate this procedure until we satisfy $\sum_{i=1}^n D_i v_i t_i \leq 19\,562$. This finally occurs when we decrease t_{21} from $\frac{1}{12}$ to $\frac{3}{52}$. Since $\sum_{i=1}^n D_i v_i t_i = 19\,554 < 19\,562$, we move to Step 5.

Step 5: Since we have \$8 remaining, we want to use them up if possible. We compute the priority ratios of the increase of the t values dynamically using Eq. (7). The priority order of the increase of the t values is as follows:

- 21, 24, 7, 19, 13, 20, 18, 15, 12, 16, 23, 22, 6, ...

Since we cannot increase t_{21} from $\frac{3}{52}$ to $\frac{1}{12}$, we increase t_{24} from $\frac{3}{52}$ to $\frac{1}{12}$. Since, we have only \$3.76 remaining budget, which cannot be used to increase any of the t_i 's, after this assignment, we stop here. Note that the priority order of the increases in the t values is not necessarily in the exact reverse order to that of Step 3 reduction of the t values.

The heuristic solution is as follows (we put the lower bound solution, an optimal solution, and the heuristic solution together in Table 3 for easier

comparison):

$$t_1 = \frac{1}{12}, t_2 = \frac{3}{12}, t_3 = \frac{2}{12}, t_4 = \frac{3}{12}, t_5 = \frac{3}{12},$$

$$t_6 = \frac{2}{12}, t_7 = \frac{3}{52}, t_8 = \frac{3}{12}, t_9 = \frac{3}{12}, t_{10} = \frac{2}{12},$$

$$t_{11} = \frac{3}{52}, t_{12} = \frac{2}{12}, t_{13} = \frac{2}{12}, t_{14} = \frac{2}{12}, t_{15} = \frac{1}{52},$$

$$t_{16} = \frac{2}{12}, t_{17} = \frac{3}{12}, t_{18} = \frac{1}{12}, t_{19} = \frac{2}{12}, t_{20} = \frac{2}{52},$$

$$t_{21} = \frac{3}{52}, t_{22} = \frac{1}{52}, t_{23} = \frac{1}{52}, t_{24} = \frac{1}{12}.$$

The objective value of the heuristic solution becomes 1582.74. Note that this solution is different from the optimal solution in two elements. Specifically, $t_{20} = \frac{2}{52}$, and $t_{21} = \frac{3}{52}$, compared to $t_{20}^* = \frac{1}{52}$ and $t_{21}^* = \frac{1}{12}$. The ratio of the objective value of the heuristic solution to that of the optimal solution is 1.0006, an extremely small cost penalty.

5. Computational studies

First, we have solved a 48 item problem of Brown [2] where it was necessary to generate the \hat{x} and σ values. We varied the size of the T set to see how the heuristic performs as the problem becomes similar to the continuous one. The excellent performance of the heuristic, at least in a limiting sense, can be confirmed as follows. The larger the size of the T set, the smaller the ratio VAL_H/VAL_{LB} , where VAL_H and VAL_{LB} are the objective function value obtained by the heuristic and the lower bound procedure, respectively. Note that we could not find an optimal solution using the branch-and-bound algorithm since the computational burden becomes enormous (Table 4).

To investigate the performance of the heuristic further, we conducted a test using randomly generated problems within realistic parameter ranges. Empirically, it has been found that quite often the distribution of usage values across a population of items can be adequately represented by a lognormal distribution [7]. Consequently, we assume that the Dv values follow such a distribution. If we let Dv

Table 4

Performance of the heuristic by varying the size of the set of time supplies

T set	VAL_H/VAL_{LB}
$T = \{\frac{1}{52}, \frac{2}{52}, \frac{3}{52}, \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}\}$	44.8036
$T = \{\frac{1}{52}, \frac{2}{52}, \frac{3}{52}, \dots, \frac{52}{52}\}$	1.9315
$T = \{\frac{1}{52}, \frac{2}{52}, \frac{3}{52}, \dots, \frac{52}{52}\} \cup \{\frac{1}{365}, \frac{2}{365}, \frac{4}{365}, \frac{8}{365}, \dots, \frac{128}{365}, \frac{256}{365}\}$	1.8449
$T = \{\frac{1}{365}, \frac{2}{365}, \frac{3}{365}, \dots, \frac{365}{365}\}$	1.0355
$T = \{\frac{1}{1000}, \frac{2}{1000}, \frac{3}{1000}, \dots, \frac{1000}{1000}\}$	1.0026

be denoted by x , then x follows the distribution

$$f(x) = \frac{1}{bx\sqrt{2\pi}} \exp\left[-\frac{(\ln x - a)^2}{2b^2}\right], \quad 0 < x < \infty,$$

where a and b are the mean and the standard deviation of the underlying normal distribution. It can be shown that the mean value of the lognormal distribution is given by

$$E(x) = \exp\left[a + \frac{b^2}{2}\right]$$

and the coefficient of variation (a measure of the relative dispersion) of the lognormal distribution is a monotonically increasing function of only b . According to Herron [8], typically the inventories of merchants (wholesalers, retailers, etc.) have b 's in the range of 0.8–2.0; industrial producers in the range 2–3; and highly sophisticated hardware suppliers (who are subject to rapid technological innovations) have b 's in the 3–4 range. Thus, we set the parameter values within these ranges. We have solved 25 randomly generated problems. Since the objective function for our problem can be transformed as follows, we only need to generate n , $D_i v_i$, $\sigma_i v_i$, $Q_i v_i$, L_i and Y parameters. We can then rewrite ETVSPY as

$$ETVSPY = \sum_{i=1}^n \frac{D_i v_i}{Q_i v_i} (\sigma_i v_i) G_u\left(\frac{D_i v_i t_i - \hat{x}_i v_i}{\sigma_i v_i}\right).$$

We have generated those parameter values as follows:

$$n \sim \text{Uniform}(15, 30), \quad L \sim \text{Uniform}(1, 13) \text{ weeks},$$

$$Dv \sim \text{LN}(7.55, 1.5^2),$$

$$\sigma v = \sqrt{L} c_1 \left(\frac{Dv}{52}\right)^{c_2} \quad \text{where } c_1 \sim \text{Uniform}(0.5, 1)$$

and $c_2 \sim \text{Uniform}(0.5, 1)$.

Since $Q_i v_i = \sqrt{2A_i D_i v_i / r_i}$, we need to generate A and r in order to generate Qv .

$$A \sim \text{Uniform}(10, 50), \quad r = 0.24/\text{year},$$

$$Y \sim \text{Uniform}(1, 2.5) \times \sum_{i=1}^n \sigma_i v_i.$$

The computational results for the 25 randomly generated problems have been summarized in Table 5. Here VAL_{OPT} represents the optimal objective value. The performance of the heuristic is reasonable. The few problems with the higher penalties (still only 5–7%) have the following common characteristic. In Step 3 of the heuristic, the budget reduction associated with the last item, whose t_i was lowered, left a relatively large unused budget (i.e. undershoot of the constraint). Then in Step 5 many t_i values had to be increased, a process which apparently does not get as close to the optimal solution as does the main part of the Heuristic (Step 3) which begins with an infeasible solution and reduces t_i 's (this is consistent with the findings of some experiments where we tried a completely

Table 5
Computational results for randomly generated problems

Mean (VAL_H/VAL_{OPT})	Maximum (VAL_H/VAL_{OPT})	Number of problems with $VAL_H = VAL_{OPT}$
1.0104	1.0678	8

different heuristic, starting from rounding *down* the lower bound solution to the nearest T_j 's. Specifically, this reverse type of heuristic did not work as well as the one described in this paper).

We did not test problems having more than 30 items because of the computational burden associated with the branch and bound algorithm for this size. However, we developed an approximate dynamic program which can be used to solve large size problems. The idea is to divide both sides of the budget constraint by a constant to reduce the number of states. Clearly, it does not necessarily produce an optimal solution. We have tested 25 randomly generated problems with n values between 50 to 100. Interestingly, our heuristic outperformed the approximate dynamic program.

6. Conclusions

In this paper we have analyzed a problem of setting reorder points (or safety stocks) of a population of items, subject to a restricted set of possible values (expressed as time supplies) as well as a limit on the total value of the safety stock. A branch-and-bound formulation for obtaining the optimal solution has been presented. In addition, a much simpler heuristic solution procedure has been shown to provide excellent results on a set of realistic examples. However, the incorporation of the pragmatic constraint of a restricted set of time supplies tends to severely deteriorate the performance (much higher ETVSPY for a given safety stock budget) compared with the unrestricted case. This is particularly true when the set of possible time supplies is fairly sparse, particularly at the lower end, which is likely to be the case in practice. Thus, the message for practitioners is clear, namely that management's desire for an easily understood and implementable reorder point policy may come with a high associated cost in terms of degradation of service. This contrasts with our earlier findings [9] for a similar problem of selecting order quantities subject to a prescribed maximum number of replenishments per year. A possible way of at least partially improving the performance for a specified total number of possible time supplies would be to

choose them so that they are more tightly packed at the lower end, such as by the so-called powers-of-two approach [10]. Specifically, one would choose the smallest time supply at a convenient value, such as 1 day, then have the higher ones at powers of two, i.e. 2, 4, 8, 16, etc., times this value. In Table 4, we can see that the performance has been slightly improved by adopting this approach.

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