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TECHNICAL NOTE

A NOTE ON LEAD TIME AND DISTRIBUTIONAL ASSUMPTIONS IN CONTINUOUS REVIEW INVENTORY MODELS

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Abstract—Scope and Purpose

There is a rapidly growing literature on modelling the effects of investment strategies to control *givens* such as setup time, setup cost, quality level and lead time. Recently, a continuous review inventory model with a mixture of backorders and lost sales in which both lead time and the order quantity are decision variables has been studied. The objectives of this paper are twofold. Firstly, we want to correct and improve the recently studied model by simultaneously optimizing both the order quantity and the reorder point. A significant amount of savings over the model can be achieved. We illustrate these savings by solving the same examples in the study. Secondly, we then develop a minimax distribution free procedure for the problem.

Recently, there have been some studies on lead time reduction to provide more meaningful mathematical models to decision makers. Ouyang *et al.* study a continuous review inventory model in which lead time is a decision variable. However, their algorithm cannot find the optimal solution due to the flaws in the modeling and the solution procedure. We present a complete procedure to find the optimal solution for the model. In addition to the above contribution, we also apply the minimax distribution free approach to the model to devise a practical procedure which can be used without specific information on demand distribution.

Key words: Inventory model, lead time reduction, continuous review, distribution free approach

1. INTRODUCTION

There are a variety of assumptions inherent in any quantitative model that need to be relaxed in a structured fashion to improve the flexibility and usefulness of the models. Silver [1] suggested a wide variety of possible improvements to undertake (equivalently, usual *givens* to change) in the operations of manufacturing, such as setup reduction, higher quality level, controllable production rates, etc. There is a rapidly growing literature on modeling the effects of changing the *givens* in the manufacturing decision situations. A detailed review up to 1997 is given by Silver *et al.* [2]. Most of the models include the cost of the change, usually amortized as part of total relevant cost per unit time.

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Recently, researchers have studied investment strategies to control *givens* such as setup time, setup cost, quality level, lead time, etc. Without the self-effort to improve the givens, there would be no success for Japanese manufacturing companies. Liao and Shyu [3] have initiated a study on lead time reduction by presenting an inventory model in which lead time is a decision variable and the order quantity is predetermined. Ben-Daya and Raouf [4] have extended the Liao and Shyu [3] model by allowing both lead time and the order quantity as decision variables. Recently, Ouyang *et al.* [5] have generalized the Ben-Daya and Raouf [4] model by allowing backorders and lost sales. However, there is a critical flaw in their model which will be explained later.

In practice, the distributional information about the demand is often limited. Sometimes all that is available is an educated guess of the mean and of the variance. There is a tendency to use the normal distribution under these conditions. However, the normal distribution does not offer the best shield against the occurrences of other distributions with the same mean and same variance. Scarf [6] addresses a newsboy problem where only the mean μ and the variance σ^2 of the demand are known without any further assumptions on the form of the distribution of the demand. Taking a conservative approach, he modeled the problem as that of finding the order quantity that maximizes the expected profit against the worst possible distribution of the demand with the mean μ and the variance σ^2. The approach is called the minimax distribution free approach. Recently, there have been several papers related to the distribution free approach which help disseminate the approach that has been calm for several decades. Gallego and Moon [7] have presented very compact proof of the optimality of Scarf's ordering rules for the newsboy problem and extended the analysis to several cases including fixed ordering cost case, multi-product case, random yield case and recourse case. Shore [8] derives explicit approximate solutions to the standard newsboy problem, to some (Q, r) models and to a periodic review model in which the first three or four moments of the demand are known. Moon and Choi [9] have applied the approach to the two-echelon stochastic production/inventory models in which assemble-to-order (ATO), assemble-to-make (ATM) and composite policies can be adopted. Gallego *et al.* [10] consider stochastic finite-horizon inventory models with discrete demand distributions that are incompletely specified by selected moments, percentiles or a combination of moments and percentiles.

The purposes of this paper are twofold. Firstly, we point out a flaw in the Ouyang *et al.* [5] model and improve their model by simultaneously optimizing both the order quantity and the reorder point. A significant amount of savings over their model can be achieved. We illustrate these savings by solving the same examples they considered. Secondly, we then develop a minimax distribution free procedure for the problem.

2. BASIC MODEL FORMULATION

We use the same notation as in Ouyang *et al.* [5] to avoid any possible confusion. However, the reorder point r, which has been assumed to be given in Ouyang *et al.* [5], becomes a decision variable. Note that the assumptions are exactly the same as those in Ouyang *et al.* [5] except the following assumption:

The reorder point r = expected demand during lead time + safety stock (SS) and SS = k (standard deviation of lead time demand), i.e. $r = \mu L + k\sigma\sqrt{L}$ where k is the safety factor. Here r is a decision variable.

They put a restriction that the reorder point must satisfy the following equation which implies a service level constraint. $P(X > r) = P(Z > k) = q$. They made a crucial mistake by including both the service level constraint and the shortage cost into the model in which both are being used redundantly to determine the appropriate level of safety stocks. We should not include the service level constraint if the shortage cost is explicitly included [2]. It is obvious that we can obtain a better solution by allowing the reorder point as a decision variable.

Since the lead time demand X follows a normal distribution with mean μL and standard deviation $\sigma\sqrt{L}$, the expected shortage at the end of the cycle is given by

$$B(r) = \int_r^\infty (x - r) f(x)\,\mathrm{d}x = \sigma\sqrt{L}\Psi(k)$$

where

$$\Psi(k) \equiv \phi(k) - k[1 - \Phi(k)]$$

and ϕ and Φ denote the standard normal probability density function and cumulative distribution function, respectively. Then, the total expected annual cost can be represented as follows.

$C(Q, r, L)$ =ordering cost + holding cost + stockout cost + lead time crashing cost

$$=\frac{AD}{Q}+h\left[\frac{Q}{2}+r-\mu L+(1-\beta)B(r)\right]+\frac{D}{Q}[\pi+\pi_0(1-\beta)]B(r)+\frac{D}{Q}R(L)$$

$$=\frac{AD}{Q}+h\left[\frac{Q}{2}+k\sigma\sqrt{L}\right]+\left\{h(1-\beta)+\frac{D}{Q}[\pi+\pi_0(1-\beta)]\right\}\sigma\sqrt{L}\Psi(k)+\frac{D}{Q}\left[c_i(L_{i-1}-L)\right.$$

$$\left.+\sum_{j=1}^{i-1}c_j(b_j-a_j)\right],\quad L\in(L_i, L_{i-1}) \tag{1}$$

We can show that equation (1) is convex with (Q, r) for a given value of L. Taking the partial derivatives of $C(Q, r, L)$ with respect to Q and r in each time interval (L_i, L_{i-1}), we obtain

$$\frac{\partial C(Q, r, L)}{\partial Q}=-\frac{AD}{Q^2}+\frac{h}{2}-\frac{D}{Q^2}[\pi+\pi_0(1-\beta)]B(r)-\frac{DR(L)}{Q^2} \tag{2}$$

$$\frac{\partial C(Q, r, L)}{\partial r}=h+h(1-\beta)(F(r)-1)+\frac{D}{Q}[\pi+\pi_0(1-\beta)](F(r)-1) \tag{3}$$

As shown by Ouyang *et al.* [5], $C(Q, r, L)$ is concave in $L \in (L_i, L_{i-1})$ for fixed (Q, r).

Therefore, for fixed (Q, r), the minimum total expected annual cost will occur at the end points of the interval. Upon setting $\partial C(Q, r, L)/\partial Q = 0$ and $\partial C(Q, r, L)/\partial r = 0$, we get

$$Q=\left[\frac{2D\{A+R(L)+[\pi+\pi_0(1-\beta)]B(r)\}}{h}\right]^{1/2} \tag{4}$$

$$1-F(r)=\frac{hQ}{hQ(1-\beta)+D[\pi+\pi_0(1-\beta)]} \tag{5}$$

The optimal (Q, r) pair given L can be obtained by solving the above Equations (4) and (5) iteratively until convergence. We can easily prove the convergence of the procedure by adopting a similar graphical technique used in Hadley and Whitin [11]. Thus, we can use the following overall procedure to find the optimal Q, r and L.

Step 1: For each L_i, $i = 1, \ldots, n$, start with

$$Q=\left[\frac{2D\{A+R(L_i)\}}{h}\right]^{1/2}$$

Repeat step 2 and step 3 until convergence.

Step 2: Find r from equation (5) using a line search.

Step 3: With r found in step 2, compute Q from equation (4).

Step 4: For each pair (Q_i, r_i, L_i), compute the corresponding total expected annual cost $C(Q_i, r_i, L_i)$, $i = 0, 1, 2, \ldots, n$. The optimal Q, r and L will be the values for which the total expected annual cost is minimum.

Example 1. In order to illustrate the above solution procedure, let us consider an inventory system with the data used in Ouyang *et al.* [5]: $D = 600$ units/year, $A = \$200$ per order, $h = \$20$ per item per year, $\pi = \$50$ per shortage, $\pi_0 = \$150$ per lost sales, $\sigma = 7$ units/week and the lead time has three components as in Table 1 in Ouyang *et al.* [5].

Table 1. Comparison of the two procedures (L_i in week)

β	Ouyang *et al.* service level	Ouyang *et al.* (Q_s, L_s)	Ouyang *et al.* $C(\cdot)$	Moon and Choi service level	Moon and Choi (Q_s, r_s, L_s)	Moon and Choi $C(\cdot)$	Percentage of savings
0.0	0.800	(177, 3)	3780.00	0.980	(121, 75, 4)	2991.85	20.9%
0.5	0.800	(158, 4)	3408.93	0.970	(121, 72, 4)	2941.68	13.7%
0.8	0.800	(144, 4)	3123.70	0.952	(121, 69, 4)	2890.56	7.5%
1.0	0.800	(134, 4)	2917.82	0.922	(122, 66, 4)	2832.00	2.9%

We solve the cases when $\beta = 0, 0.5, 0.8$ and 1. Note that Ouyang *et al.* [5] set the service level to 80% and consequently, the reorder point has been predetermined. We summarize the computational results in Table 1. The savings range from 2.9 to 20.9% which shows significant savings can be achieved by simultaneously optimizing over both the order quantity and the reorder point. Note that the savings increase as β decreases. It is interesting to observe that our procedure results in a higher service level for every case by spending less money.

3. DISTRIBUTION FREE APPROACH

Now, we consider the distribution free approach. We make no assumption on the distribution F of X other than saying that it belongs to the class $\mathscr{F}$ of cumulative distribution functions with mean μ and variance σ^2. Since the distribution F of X is unknown we want to minimize the total expected annual cost against the worst possible distribution in $\mathscr{F}$. We can represent the total expected annual cost as follows:

$$C(Q, r, L) = \frac{AD}{Q} + h\left[\frac{Q}{2} + r - \mu L\right] + \left\{h(1-\beta) + \frac{D}{Q}[\pi + \pi_0(1-\beta)]\right\} E[X-r]^+ + \frac{D}{Q}\left[c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)\right], \quad L \in (L_i, L_{i-1})$$

where we let $x^+ = \max\{x, 0\}$.

The distribution free approach for this model is to find the most unfavourable distribution in F for each (Q, r, L). Our problem is to solve:

$$\min_{Q>0,\ r>\mu L,\ L>0} \max_{F\in\mathscr{F}} C(Q, r, L)$$

To this end, we need to use the following proposition as in Gallego and Moon [7]:

Proposition 1. *For any* $F \in \mathscr{F}$

$$E[X-r]^+ \leq \tfrac{1}{2}\{\sqrt{\sigma^2 L + (r-\mu L)^2} - (r - \mu L)\} \tag{6}$$

Moreover, the upper bound, equation (6), is tight. In other words, we can always find a distribution in which the above bound is satisfied with equality for every r.

Using Proposition 1, our problem is to minimize the cost function for the worst distribution

$$C^{W}(Q, r, L) = \frac{AD}{Q} + h\left[\frac{Q}{2} + r - \mu L\right] + \frac{D}{Q}\left[c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)\right] + \left\{h(1-\beta) + \frac{D}{Q}[\pi + \pi_0(1-\beta)]\right\} \frac{\sqrt{\sigma^2 L + (r-\mu L)^2} - (r-\mu L)}{2} \quad L \in (L_i, L_{i-1}) \tag{7}$$

We can show that equation (7) is convex with (Q, r) for a given value of L. Taking the partial derivatives of $C^{W}(Q, r, L)$ with respect Q and r in each time interval (L_i, L_{i-1}), we obtain

$$\frac{\partial C^{W}(Q, r, L)}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{D[\pi + \pi_0(1-\beta)]}{2Q^2}[\sqrt{\sigma^2 L + (r-\mu L)^2} - (r-\mu L)] - \frac{D}{Q^2} R(L) \tag{8}$$

$$\frac{\partial C^{W}(Q,r,L)}{\partial r}=h+\left[h(1-\beta)+\frac{[\pi+\pi_0(1-\beta)]D}{Q}\right]\frac{1}{2}\left\{\frac{r-\mu L}{\sqrt{\sigma^2L+(r-\mu L)^2}}-1\right\} \tag{9}$$

$C^{W}(Q, r, L)$ is concave in $L\in(L_i, L_{i-1})$ for fixed (Q, r), because

$$\frac{\partial^2 C^{W}(Q,r,L)}{\partial L^2}=-\frac{1}{4}hk\sigma L^{-3/2}-\frac{1}{8}\left\{h(1-\beta)+\left[\pi+\pi_0(1-\beta)\frac{D}{Q}\right]\right\}\sigma L^{-3/2}(\sqrt{1+k^2}-k)<0$$

where $k = (r-\mu L)/\sigma\sqrt{L}$

Therefore, for fixed (Q, r), the minimum total expected annual cost with worst distribution will occur at the end points of the interval. Upon setting $\partial C^{W}(Q, r, L)/\partial Q = 0$ and $\partial C^{W}(Q, r, L)/\partial r = 0$, we get

$$Q=\left[\frac{D\{2A+[\pi+\pi_0(1-\beta)][\sqrt{\sigma^2L+(r-\mu L)^2}-(r-\mu L)]+2R(L)\}}{h}\right]^{1/2} \tag{10}$$

$$\frac{r-\mu L}{\sqrt{\sigma^2L+(r-\mu L)^2}}=1-\frac{2hQ}{hQ(1-\beta)+D[\pi+\pi_0(1-\beta)]} \tag{11}$$

The optimal (Q, r) pair given L can be obtained by solving Equations (10) and (11) iteratively until convergence. The convergence of the procedure can be shown. Moreover, we can use the similar procedure used in the previous section to find the optimal Q, r, L and denote it as (Q^{W}, r^{W}, L^{W}).

Example 2. We use the same data as in Example 1. Note that the mean and the standard deviation of the demand are all the information that we can obtain. We compare the performance of (Q^{W}, r^{W}, L^{W}) with (Q^{N}, r^{N}, L^{N}) where $N\in\mathscr{F}$ represents the normal distribution. The results are $(Q^{W}, r^{W}, L^{W}) = (152, 57, 3)$ and $(Q^{N}, r^{N}, L^{N}) = (121, 69, 4)$, and the worst case annual expected cost $C^{W}(Q^{W}, r^{W}, L^{W})$ is \$3,474.86 for the $\beta = 0.8$ case. The cost of using (Q^{W}, r^{W}, L^{W}) instead of the optimal (Q^{N}, r^{N}, L^{N}) for a normal distribution is clearly

$$C^{N}(Q^{W}, r^{W}, L^{W})-C^{N}(Q^{N}, r^{N}, L^{N})=\$3{,}027.44-\$2{,}890.56=\$136.88$$

Here $C^{N}(Q^{W}, r^{W}, L^{W})$ is the annual expected cost of using (Q^{W}, r^{W}, L^{W}) when the actual demand distribution is normal. This is the largest amount that we would be willing to pay for the knowledge of F. This quantity can be regarded as the Expected value of additional information (EVAI) [8].

We solve the cases when $\beta = 0$, 0.5, 0.8 and 1 as in Example 1. From the results of the above example as well as Table 2, we can reconfirm the robustness of the distribution free approach which has been widely proven in recent studies [7, 9, 10].

4. CONCLUDING REMARKS

We have presented a continuous review inventory model with a mixture of backorders and lost sales in which the order quantity, the reorder point and lead time are decision variables. This model improves the existing one, and results in both significant savings in the total expected

Table 2. Computational results

β	(Q^{W}, r^{W}, L^{W})	$C^{W}(Q^{W}, r^{W}, L^{W})$	$C^{N}(Q^{W}, r^{W}, L^{W})$	$C^{N}(Q^{N}, r^{N}, L^{N})$	$C^{N}(Q^{W}, r^{W}, L^{W})/C^{N}(Q^{N}, r^{N}, L^{N})$
0.0	(166, 70, 3)	4048.20	3303.64	2991.85	1.104
0.5	(158, 63, 3)	3726.30	3137.05	2941.68	1.066
0.8	(152, 57, 3)	3474.86	3027.44	2890.56	1.047
1.0	(142, 66, 4)	3225.61	2859.32	2832.00	1.010

annual cost and higher service level. We have also applied the distribution free approach to the basic model. One interesting research area is to apply the lead time reduction concept to other inventory models to justify the investment to reduce the lead times. The piece-wise linear crashing cost function is widely used in project management in which the duration of some activities can be reduced by assigning more resources to the activities. One might consider using other types of crashing cost function as in Ben-Daya and Raouf [4] if the piece-wise linear function is not appropriate to reflect the crashing relation.

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