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Warranty cost analysis under continuous sales

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The modeling and analysis of a variety of warranty policies and some related optimisation problems have been widely studied during the last decades. Recently, Blischke and Murthy proposed a new model for warranty servicing under continuous sales. However, we point out that there is a crucial mistake in their formulation, and provide a correct formulation. We illustrate the amounts of estimation errors using the numerical example of Blischke and Murthy.

Keywords: warranty reserve; pro-rata warranty; continuous sales

Introduction

The primary role of a warranty is to ensure a postpurchase remedy for consumers. It offers protection when an item, properly used, fails to perform as intended or as specified by the manufacturer. When a product is sold with a warranty, the manufacturer has to observe the warranty contract. The manufacturer provides a refund, or replaces a failed product with a new one in the case of nonrepairable products, or repairs a failed product until the warranty period expires. These actions are called warranty servicing.[1] In general, consumers like long warranty periods during which they can use a product without experiencing costs from its failure. Thus, manufacturers use warranty policies as promotional tools in their marketing strategy. The longer the warranty period, the more competitive the product. Extensive warranties, however, add costs to the manufacturer. For this reason, warranty planning is an important decision problem.

The manufacturer should estimate future warranty costs, and many studies have been done to investigate warranty costs. Excellent reviews up to 1996 have been found in both Murthy and Blischke[2] and Blischke and Murthy.[3] Most of works have dealt with warranty servicing for a single item, using simple service strategies, either always repair or always replace. In addition, most of the existing warranty cost models assume single lot (batch) sales. Menke[4] has studied the pro-rata warranty reserving problem under single lot sales. Amato and Anderson[5] have extended the model of Menke[4] to allow for discounting and price level changes over time. Thomas[6] has generalised the results of Amato and Anderson[5] to the case of a general product failure distribution.

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If the manufacturer offers an unconditional refund when nonrepairable items fail under warranty, the manufacturer must set aside a fraction of the revenue generated by sales. This is called warranty reserving and models for both single lot sales and continuous sales over the product life cycle have been provided by Blischke and Murthy.[1,3] However, there is a crucial mistake in their formulation for continuous sales due to an inappropriate relationship between the sales timing and the warranty period. Here we explain the mistake, and develop a new formulation to correct their mistake. The differences between their model and ours are illustrated by the same example they used.

Review and correction of the model by Blischke and Murthy

We first review the model developed by Blischke and Murthy.[1,3] The same notation is used to avoid any possible confusion.

Notation

$F(t)$: distribution function for the first time to failure
$f(t)$: density function for the first time to failure
W: warranty period
L: product life cycle, that is sales occur over the interval $[0, L]$
$s(t)$: sales rate (that is, sales per unit time) over the life cycle, $0 \leqslant t \leqslant L$
$v(t)$: refund rate at time t
c_b: purchase price of the product

We consider nonrepairable product sold with a nonrenewable pro-rata warranty policy.

Under this policy the manufacturer agrees to refund a fraction of the purchase price should the product fail before time W from the time of the initial purchase. The buyer is not constrained to buy a replacement product. We assume that continuous sales occur during the life cycle, and products are put into use immediately after they are purchased. We also assume that the life cycle L exceeds W, the warranty period. Since the manufacturer must provide a refund for products that fail before reaching age W, and since the last sale occurs at or before time L, the manufacturer has an obligation to service warranty claims over the interval $[0, L+W]$. Note that the manufacturer has to provide a refund only if the failed product is of age less than W.

Let Y be the refund for the failed product. We assume that the refund is a linear function depending on the age of the product at failure (X) as follows.

$$Y = \begin{cases} c_b(1 - X/W) & \text{if } X < W \\ 0 & \text{if } X \geqslant W \end{cases} \tag{1}$$

Let S denote the total sales over the life cycle. Then S is given by

$$S = \int_0^L s(t)dt \tag{2}$$

The refund rate (that is, amount refunded per unit time), $v(t)$, is a random variable, since product failures and the resulting claims occur randomly. Once we derive an expression for $v(t), 0 \leqslant t \leqslant L+W$, then the expected refund amount over the product life cycle can be determined as

$$\int_0^{L+W} v(t)dt \tag{3}$$

Note that refunds over the interval $[t, t+\delta t]$ occur due to failure of products sold over the period $[\psi, t]$ where ψ is given by

$$\psi = \max(0, t-W) \tag{4}$$

A product sold at time $t-\tau$ will fail in the interval $[t, t+\delta t]$ with probability $f(\tau)\delta t$. Since the sales rate is given by $s(t)$, we have the expected refund over $[t, t+\delta t]$ given by

$$v(t)\delta t = c_b\left\{\int_\psi^t s(\tau)f(t-\tau)\left(1-\frac{t-\tau}{W}\right)d\tau\right\}\delta t \tag{5}$$

Blischke and Murthy[1,3] have used the above equation to derive the following equation. Note that they write $(t-\tau)/W$ instead of $(1-(t-\tau)/W)$ due to an inadvertent error

$$v(t) = c_b\left\{\int_\psi^t s(\tau)f(t-\tau)\left(1-\frac{t-\tau}{W}\right)d\tau\right\},\quad 0 \leqslant t \leqslant L+W \tag{6}$$

Using (4), (6) can be rewritten as follows:

$$v(t) = \begin{cases} c_b\left\{\int_0^t s(\tau)f(t-\tau)\left(1-\frac{t-\tau}{W}\right)d\tau\right\}, & \text{if } 0 \leqslant t \leqslant W \\ c_b\left\{\int_{t-W}^t s(\tau)f(t-\tau)\left(1-\frac{t-\tau}{W}\right)d\tau\right\}, & \text{if } W < t \leqslant L+W \end{cases} \tag{7}$$

The above equation contains a critical mistake which will overestimate the total amount of the refund. Since the last sale occurs at or before time L, we only need to consider the sales rate up to time L to compute the refund rate for $t > L$. In other words, the above equation needs to be divided into three parts rather than two parts to properly compute the appropriate amounts of refund rate as follows:

$$v(t) = \begin{cases} c_b\left\{\int_0^t s(\tau)f(t-\tau)\left(1-\frac{t-\tau}{W}\right)d\tau\right\}, & \text{if } 0 \leqslant t \leqslant W \\ c_b\left\{\int_{t-W}^t s(\tau)f(t-\tau)\left(1-\frac{t-\tau}{W}\right)d\tau\right\}, & \text{if } W < t \leqslant L \\ c_b\left\{\int_{t-W}^L s(\tau)f(t-\tau)\left(1-\frac{t-\tau}{W}\right)d\tau\right\}, & \text{if } L < t \leqslant L+W \end{cases} \tag{8}$$

We illustrate the size of errors caused from the above mistake by solving the example used in Blischke and Murthy.[1,3]

Example Let $L = 4$ years and suppose that the sales rate over the product life cycle is given by $s(t) = kt \cdot \exp(-t)$, $0 \leqslant t \leqslant 4$ with $k = 1100$. This value was chosen to yield total sales of $S = 1000$. Let $F(t)$ be an exponential distribution with parameter $\lambda = 1$. Thus, the mean age of items at failure is one year. Let $W = 1$ year and $c_b = \$10$.

The expected refund rate $v(t)$ using the equations of Blischke and Murthy[1,3] is as follows (Note that the following equations are slightly different from those in Blischke and Murthy[1,3] due to the inadvertent error in writing (6)):

$$v(t) = \begin{cases} \dfrac{c_b k e^{-t} t^2(3-t)}{6} & \text{if } 0 \leqslant t \leqslant 1 \\ \dfrac{c_b k e^{-t}(3t-1)}{6} & \text{if } 1 < t \leqslant 5 \end{cases}$$

Then the expected refund amount over the product life cycle is

$$\int_0^{L+W} v(t)dt = \frac{c_b k}{6}\left[\int_0^1 e^{-t}t^2(3-t)dt + \int_1^5 e^{-t}(3t-1)dt\right]$$
$$= \frac{c_b k}{6}[6e^{-1} - 17e^{-5}] = \$3837$$

The expected refund rate $v(t)$ using our correct equations is as follows:

$$v(t) = \begin{cases} \dfrac{c_b k e^{-t} t^2(3-t)}{6} & \text{if } 0 \leqslant t \leqslant 1 \\ \dfrac{c_b k e^{-t}(3t-1)}{6} & \text{if } 1 < t \leqslant 4 \\ \dfrac{c_b k e^{-t}}{6}[L^2(3-3t+2L) - (t-W)^2(3-t-2W)] & \\ & \text{if } 4 < t \leqslant 5 \end{cases}$$

Then the expected refund amount over the product life cycle is

$$\int_0^{L+W} v(t)dt = \frac{c_b k}{6}\left\{\int_0^1 e^{-t}t^2(3-t)dt + \int_1^4 e^{-t}(3t-1)dt + \int_4^5 e^{-t}[L^2(3-3t+2L) - (t-W)^2(3-t-2W)]\right\}dt = \$3676$$

Note that the expected refund amount using the equations from Blischke and Murthy[1,3] overestimates by 4.4% compared to the true expected refund amount.

Remark Let γ denote the fraction of the sale price set aside to form the warranty reserve, and suppose that it is selected so that the input to warranty reserves over the product life cycle equals the total expected payout. This implies that

$$\gamma = \frac{\int_0^{L+W} v(t)dt}{c_b S} \tag{9}$$

The following equation is shown in Blischke and Murthy[1,3]

$$\gamma = \frac{\mu(W)}{c_b} \tag{10}$$

where

$$\mu(W) = c_b \int_0^W \left(1 - \frac{x}{W}\right) f(x)dx$$

is the mean of the refund for the failed product. It can be easily shown that $\mu(W)$ is \$3.676 for the above example. Consequently, we can confirm that our derivation is correct using (9) and (10) (That is, $\gamma = 0.3676$ from our deviation, is equal to $\mu(W)/c_b$ from (10)).

Remark The crucial mistake of Blischke and Murthy[1,3] has been repeated in the other models for continuous sales as follows:

(i) When they compute the total expected discounted cost to service the warranty, the incorrect $v(t)$ has been used.

(ii) When a nonrepairable product is solid with a nonrenewing free-replacement policy, the manufacturer has to replace all products that fail within warranty period W. The expected demand rate for spares at time t, $\rho(t)$ is given by

$$\rho(t) = \int_\psi^t s(\tau)m(t-\tau)d\tau$$

where $m(t)$ is the renewal density function. Using the definition of ψ, $\rho(t)$ can be divided into two cases. However, $\rho(t)$ must be divided into three cases as explained before, since the last sale occurs at or before time L.

(iii) Suppose that failed products are repairable and they are sold with a free-replacement warrant policy. We assume that whenever a failed product is returned under warranty, the manufacturer always repairs the failed product and returns it to the owner. We consider the minimal repair case where the failure rate of the product after repair is the same as that just before failure. The expected return rate for repair, $\rho_t(t)$, is given by

$$\rho_r(t) = \int_\psi^t s(\tau)r(t-\tau)d\tau$$

where $r(t)$ is the failure rate function associated with the failure distribution function $F(t)$. Using the definition of ψ, $\rho_r(t)$ is divided incorrectly into two cases.

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