Distributionally Robust Multiperiod Inventory Model for Omnichannel Retailing Considering Buy-Online, Pickup-in-Store and Out-of-Stock, Home-Delivery Services

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***Abstract*—This article proposes the distributionally robust multiperiod inventory model incorporating the buy-online-pickup-in-store (BOPIS) and out-of-stock-home-delivery-service (OSHDS), which are the representative services of omnichannel retailing. Under this omnichannel system, the retailer operates both online and brick-and-mortar (B&M) stores simultaneously, which allow interactive flows of customer demands and desired products. The BOPIS allows customers who buy products through the online store to pick them up in the B&M store. Meanwhile, the OSHDS allows customers who find the product they want out of stock in a B&M store to receive it later, through express delivery from the online store. To capture the correlated uncertain demands of the BOPIS and OSHDS, we adopt a factor-based demand model that is affinely dependent on predefined uncertain factors. To handle a multistage decision process under uncertain demands, we utilize a rule-based approximation and distributionally robust bound to derive a tractable formulation. Computational results achieved in this article offer some insights that BOPIS and OSHDS play a role in providing retailer with flexibility, which could handle unsatisfied demands efficiently.**

***Index Terms*—Buy-online-pickup-in-store (BOPIS), distributionally robust optimization, inventory model, omnichannel retailing, out-of-stock-home-delivery-service (OSHDS).**

I. Introduction

With the prevalent use of mobile devices, such as smartphones and tablet computers, significant changes have occurred in the distribution industry [1]. The distinctive feature of this, as it relates to contemporary retailing, is the "anytime, anywhere" environment customers have come to expect. Using this platform to shop, customers interact with multiple channels, including the online and offline channels, and cross them interchangeably to suit their demands [2]. For retailers, this phenomenon emphasizes the necessity to find desired products easily across channels by integrating channels accordingly [3]. *Omnichannel* retailing is one of the strategies that companies utilize in this technical process. It is a compound word from the Latin word *omni*, which means "everything" or "universal," and *channel*, which refers to the distribution channel of products [4]. As can be inferred from this compound word, the omnichannel retailing allows customers to purchase products through every channel available, including online and offline methods, from anywhere in the world. By combining each distribution channel's characteristics, customers can feel that they are shopping at the same store on any channel at the time, regardless of the type of channel they use [5].

The distribution industry originated from a *single-channel* system, consisting of only one channel, such as the traditional brick-and-mortar (B&M) store. As access to and use of e-commerce became more widespread, retailers began operating in a *multichannel* way [6]. This switch could acquire new customers who could not be accessed through the single channel by offering various options and conveniences [7]. Although the multichannel marketing method has a stronghold on managing customers across various channels, product management and organizational management are still separate for each channel, which does not create a seamless interface across multiple touchpoints. There even have been competition within the same company, which results in less efficient operation than an integrated system offers [8], [9]. The introduction of a cross-channel could compensate for the shortcomings of a multichannel by alleviating the competition among the separated channels that have not achieved substantial interchannel integration. An elegant solution to this problem was the introduction of the omnichannel system, which is an integrated system that provides a seamless shopping experience to customers [10].

The term omnichannel was first coined by the *Harvard Business Review* in 2011, by replacing the words "multichannel" or "cross-channel," and has become the chosen word in the distribution industry. The omnichannel system also has been adopted as a survival strategy for retailers at traditional B&M

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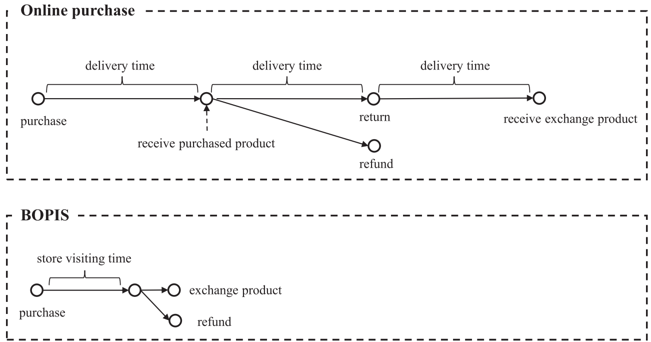

Fig. 1. Comparison of online purchase and BOPIS with regard to delivery time.

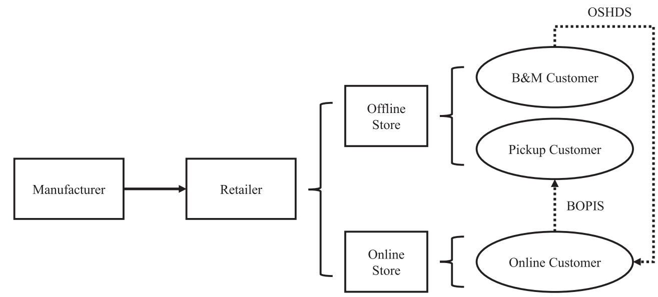

Fig. 2. Framework of the supply chain system considering BOPIS and OSHDS in an omnichannel system.

stores in the face of strong mobile shopping trends [11], [12]. In essence, the concentration of online stores caused the crisis of offline stores. To overcome this crisis, companies introduced the omnichannel, maintaining the advantages of an online store while also utilizing the advantages of an offline store [13]. Maintaining the offline store in the omnichannel system could reduce product uncertainty among customers who want to purchase through the online store [14]. An omnichannel system allows customers to search for a product at any time, subsequently having their products delivered whenever they choose [15].

Buy-online-pickup-in-store (BOPIS) is one of the representative services of the omnichannel system operated by major multinational technology company, such as *Apple*, or worldwide clothing companies, such as *Gap, Banana Republic, Old Navy, and Athleta.* Xu *et al.* [16] demonstrated that the retailer will maintain or lower the price after implementing the BOPIS. That is, with this omnichannel service, customers can purchase the product at a low price or benefit from the product's promotion through an online store and pickup the product at an offline store, with no delivery time. Furthermore, this service could be beneficial for customers who buy the product through an online store but then want to return their products to get a different size or to get a refund because of dissatisfaction with the product [17]. By exchanging or refunding a product immediately, customers could save on the delivery time when compared with the online purchasing process. Comparison between online purchase and BOPIS concerning a return or exchange process is illustrated in Fig. 1. Meanwhile, *out-of-stock-home-delivery-service* (OSHDS) was introduced, which has the opposite direction of the flow of the BOPIS option. This service could prevent potential lost sales when a product is out of stock by delivering the product directly to the place where the customer wants to receive it. Hence, the potential can be minimized, and backlogs can be treated efficiently. Although express delivery incurs additional costs, the system of ordering this way can nevertheless prevent lost sales. With this omnichannel distribution setup, customers can complete a purchase and receive orders even when the pursued product is out of stock. Because of the two omnichannel services, customers can simultaneously use various channels in their shopping process, start their search using one channel, and finish their purchasing in another. Due to the wave of omnichannel distribution, customizing delivery choices based on the preference of each customer can now become practicable [13], [15], [18]. Moreover, customers have more chances to try out products for themselves before purchase [19].

From the retailer's perspective, centralized inventory management through an omnichannel system can lower the amount of on-hand inventory while increasing stocks available to meet demand and acquire higher benefits of the omnichannel strategy [13], [20]. However, adopting the omnichannel system complicates inventory control [21], [22]. When it comes to customer demand in the multichannel system, each channel is responsible for its demand and fulfillment [23]. That is, the demand in one channel does not affect the demand in another channel, which can be handled independently [24]. Accordingly, inventory managers exist for each channel, and each demand is considered an independent demand. On the contrary, in the omnichannel system, demand caught on one side can affect the inventory level on the other side. When the demand within each channel follows uncertainty, the dependency or correlation among each channel should be considered [25], [26]. That is, demand modeling in the omnichannel would be more complicated than it is in the multichannel. As the omnichannel system became the new norm in the distribution sector, the traditional inventory control method reached the limit [27]. Before the emergence of omnichannel system, the demand and information flows were predictable in the whole supply chain system. Unlike how they behaved within this traditional supply chain, the demand and product in the omnichannel now flow in all directions. Customers increasingly expect omnichannel fulfillment with browsing on their smartphones. They also expect to be able to make purchases online for in-home delivery. This expectation has hastened the redesign of inventory and supply chain systems [28].

This article considers BOPIS and OSHDS from the perspective of the multiperiod inventory problem. We assume that two types of directional flows exist between online and offline stores. In the case of BOPIS, flow occurs from the online store to the offline store, and this direction is vice versa for OSHDS. Since all operations are made in the omnichannel system, we assume that all series of processes are conducted by one retailer. Also, to differentiate between the operation of a multichannel system, we assume that all demands are not independently handled but correlated. We illustrate the flows of the BOPIS and OSHDS in Fig. 2. As shown in Fig. 2, we restrict the operation of the omnichannel system to one manufacturer and one retailer who operates two types of inventories for the offline and online stores. The contributions of our article apart from previous studies are as follows.

1) We incorporated the BOPIS and OSHDS in the multiperiod inventory model, which are the representative services of the omnichannel system. Most previous studies have dealt with an analytical model to determine the impact of omnichannel retailing on the price of a product. However, this article covered the omnichannel retailing in an inventory model, which has mostly not been considered.
2) The omnichannel system is operated in an integrated manner, unlike the multichannel system that operates each channel independently. We characterized the correlation of the uncertain demand to incorporate the omnichannel system's distinctive feature from the perspective of demand modeling.
3) The distributionally robust optimization approach was utilized to ensure tractability in the mathematical model under the correlated uncertain demand. We derived a tractable model that features a second-order cone optimization model that can be solved with a commercial solver. By conducting various experiments, we verified the tractability of the model and proposed managerial insights.

The rest of this article is organized as follows. Previous relevant studies are investigated in Section II. In Section III, we describe the inventory model considering BOPIS and OSHDS. Section IV presents the mathematical formulation and distributionally robust optimization approach. In Section V, we present and analyze the computational results. Finally, Section VI provides a summary and critique of the findings, and includes a discussion of the implications of the findings to future research in this area. Throughout the article, we will use bold characters to denote vectors, such as $\mathbf{x}$. For the expression of the planning horizon, $t \in \mathfrak{T}$ represents $t \in \{1, \ldots, T-1\}$ and $t \in \mathfrak{T}^+$ represents $t \in \{1, \ldots, T\}$. The operators $(\cdot)^+$ and $(\cdot)^-$ mean $\max(\cdot, 0)$ and $-\min(\cdot, 0)$, respectively. Meanwhile, uncertain values, including random variables, will be denoted by the tilde, such as $\tilde{x}$.

II. Literature Review

In Section II-A, we investigated previous studies which considered the omnichannel system from the perspective of inventory problem. In Section II-B, we examined previous studies related to the (distributionally) robust optimization approach to an inventory problem, which is closely related to our problem. After giving a detailed account of the previous studies, we summarized distinguishable features of this article from previous studies.

A. Omnichannel Retailing

As the omnichannel retailing became a growing trend in the retail sector, it also gained great interest from academia, including operations management, supply chain management, and inventory management [29], [30]. Gallino and Moreno [31] studied the impact of sales of online store and offline store by introducing the BOPIS concept. They interpreted the results of the experiment with a *cross-selling effect* where customers visited an offline store to receive products purchased through an online store. Also, they analyzed a channel-shift effect where customers researched the products through the online store and then purchased them at an offline store. Gao and Su [20] analyzed the effect of BOPIS from the information and convenience point of view and how these two factors affect customers. The results addressed that the BOPIS might be useful in attracting customers, but it may cause unintended consequences by reducing store traffic. Choi *et al.* [32] analyzed the contract type and order point when a fashion company runs a franchise based on the omnichannel system. They considered a situation where the brand owner began selling to the franchisee's offline store in the first period and then to the online store in the second period. They derived several insights by analyzing the closed-form solution of the mathematical model. Zhang *et al.* [33] conducted a comparative analysis of the difference between the omnichannel strategy and the online store by considering the customer return and cancellation. They provided the optimal price and inventory decision through a mathematical model. They found that the omnichannel system is more efficient than an online store in terms of item pricing, returns and cancellations, and inventory control.

Most studies considering the omnichannel have only focused on the single-period problem. Few researchers have addressed the multistage decision problem, but they have only focused on the deterministic demand or price-dependent demand. Gupta *et al.* [34] considered the omnichannel system in which the retailer operates both multiple physical stores and one online store. They developed a multiperiod mathematical model to obtain the optimal price based on the price-sensitive demand model and proposed a heuristic algorithm to overcome the high complexity of the proposed mathematical model. Zhang *et al.* [35] also conducted a study on fulfillment within the BOPIS. They defined an in-store picking problem for BOPIS and identified the best performance frontier. MacCarthy *et al.* [36] studied the process of a retailer receiving a preorder through the online store and then allowing for customer pickup at an offline store. They analyzed conditions for the best strategy choice of this process based on a game theory approach. However, this article dealt with retailing services within a dual channel, where each store competes, which is different from the BOPIS. Bayram and Cesaret [19] considered a ship-from-store service model in which orders were placed through an online store, and products were delivered from the nearest offline store. They developed a dynamic programming formulation to derive the retailer's optimal inventory order policy. To verify the performance of the formulation, they conducted a comparative analysis with the order policy in an independently operated channel. In the case of the dynamic programming formulation, the tractability could not be guaranteed when the data size increased. They suggested a heuristic methodology that could solve the problem efficiently in terms of computation time. Shi *et al.* [37] also considered preordering within the BOPIS with both informed and uninformed customers. They assumed that the retailer sold the products to the informed customers by allowing them to preorder in the first period. Then, in the second period, products were picked up by the customer, who understood the value of the products ordered. They concluded that a preorder is not always advantageous to retailers. Furthermore, they defined a profitable condition for the retailer and argued that as the proportion of informed customers increased, the condition would become more profitable. Meanwhile, Jin *et al.* [17] considered unreasonable recommendations to customers by stores, which could be a shortcoming of the BOPIS in the omnichannel system. They also regarded the ordering option that allows customers to reserve an item through an online store and then pick it up and pay for it in an offline store. By developing a theoretical model, they derived the optimal adoption strategy for this service. Their comparative analysis concluded that BOPIS is less profitable for retailers with both liberal and strict cancellation policies, but allows retailers with moderate cancellation policies. Jain *et al.* [38] considered the facility selection model for BOPIS by utilizing the K-means clustering. By using their analytical model, they were able to evaluate each zone based on its demand, inventory carrying capacity, cost, and population characteristics. In addition, they conducted case studies that may aid understanding the selection of BOPIS facilities for omnichannel services. Gupta *et al.* [39] considered the multiperiod Stackelberg game between an omnichannel retailer and a manufacturer to describe joint production, pricing, and inventory control. By formulating the leader–follower game based on the mixed-integer nonlinear bilevel optimization problem, the manufacturer and retailer could pursue their respective maximized profits. By conducting numerical experiments, they could derive some managerial insights including that a drop-shipping channel could serve as an omnichannel characteristic but also assist in generating a higher profit. They developed a leader–follower game based on the mixed-integer nonlinear bilevel optimization problem that allows manufacturers and retailers to maximize their profits. Performing numerical experiments provided some valuable insights into management, including that drop shipping could serve as an omnichannel characteristic, while helping to generate higher profits.

Various studies have considered the omnichannel system from the perspective of inventory management and supply chain management. However, the operation of BOPIS or OSHDS was not considered in the multiperiod inventory model. Also, there have been no studies dealing with uncertain parameters for the inventory model for the omnichannel system.

B. Optimization Problem Under an Uncertain Environment

In the optimization field, *stochastic programming and robust optimization* are representative methodologies for dealing an uncertain environment. *Fuzzy theory* is another optimization approach with the uncertainty, but it mainly entails ambiguous subjective decisions rather than uncertainty of the input parameter. In this article, we consider the uncertain demand in the optimization model. That is, the literature review related to the fuzzy theory is beyond the scope of this article. Thus, we narrow the scope of the literature review on the optimization model with the uncertain input parameter. In the case of the stochastic program, the probability distribution of uncertain data is assumed to be known or estimated [40]. The problem seeks the decision variable that minimizes or maximizes the expectation of the objective function under a given probability distribution. On the other hand, in the case of the robust optimization, it is assumed that uncertain data belongs to a specific uncertain set, such as a box, ellipsoid, or polyhedron set, instead of estimating the probability distribution. Unlike the stochastic program that makes decisions about distribution, the robust optimization approach finds the optimal solution against the worst-case scenario, which could be feasible for all possible realization of scenarios. The robust optimization approach has the advantage in that it can retain the tractability from the primal deterministic formulation and could solve the problem without necessitating an estimation of the full information of the distribution. However, because of the conservative solution it mandates, it provides a worse solution than does stochastic programming. In the case of stochastic programming, computational tractability is generally restricted, except in certain conditions.

Meanwhile, the *distributionally robust optimization* approach generalizes stochastic programming and robust optimization. This approach finds the optimal solution based on the worst-case distribution or worst-case expectation under an *ambiguity set,* which contains the true distribution, but the distribution is unknown. If the candidate distribution in the ambiguity set contains only the true distribution, the problem becomes a stochastic program. On the other hand, if the problem considers all distributions under the given support set, the problem becomes a robust optimization [41], that is, the distributionally robust optimization approach is a general form of the stochastic programming and robust optimization, which provides a less conservative solution than robust optimization does, while retaining tractability from the primal deterministic formulation. This article considers the inventory policy of a retailer, based on the multistage decision process with a correlated uncertain demand. By utilizing the distributionally robust optimization approach, correlated uncertain demands could be characterized by the ambiguity uncertain factor, which only requires partial information of distribution. Accordingly, we intensively survey the literature which is relevant to (distributionally) robust optimization. Robust optimization began with the work in [42]. By defining the uncertainty set as a box shape, full perturbation of uncertain data was allowed. It is a monumental article that marks the beginning of robust optimization, but it has the weakness of being too conservative. Robust optimization came into the spotlight again from the reformulation of uncertain sets of ellipsoid and polyhedron sets to provide less conservative robust counterparts [43]–[45]. These uncertain sets were able to reformulate robust counterparts with tractable second-order cone programs and linear programs, respectively. This methodology has provided the optimal solution even for relatively large data inputs, in part because of recent developments in computing power and commercial solvers dedicated to optimization. However, these methodologies have a limitation in that they must make a decision before all uncertain data are realized, which is not suitable for a multistage decision process. The development of robust optimization based on the multistage

TABLE I
DISTINGUISHING FACTORS OF OUR RESEARCH FROM THE PREVIOUS RELEVANT RESEARCH

Previous research	**Omnichannel system**	**Time period**	**Demand modelling**	**Solution methodology**
[34]	Online and offline channel	Discrete	Latent class model	Evolutionary algorithm
[19]	Ship from store	Discrete	Poisson process	Markov decision process
[17]	BOPIS and ROPS[1]	Continuous	Expected demand	Closed-form expression
[36]	POPU[2]	Single-period	Utility function	Nash equilibrium
[35]	BOPIS	Continuous	Poisson process	Best performance frontier
[20]	BOPIS	Single period	General continuous distribution	Closed-form expression
[32]	Online and offline channel	Single period	Normal distribution	Closed-form expression
[33]	ROPS	Single period	General continuous distribution	Closed-form expression
[60]	SFS[3]	Multi period	Stochastic demand	TRO[4] and LDR[5]
[59]	BOPIS	Single period	Stochastic demand	DRO
This research	BOPIS and OSHDS	Discrete	Factor-based demand	DRO and LDR

[1] ROPS: reserve-online-pick-up-and-pay-in-store.
[2] POPU: preorder-online, pickup-in-store.
[3] SFS: ship-from-store.
[4] TRO: target-oriented robust optimization.
[5] LDR: linear decision rule.

decision started with introducing the linear (affine) decision rule by [46]. This methodology has changed decision-making from the *wait-and-see* decision to the *here-and-now* decision, which considers the variables as two types, including *adjustable* and *nonadjustable* variables. Unfortunately, an adjustable robust counterpart is generally computationally intractable. To tackle this limitation, Tal *et al.* [46] adopted the linear decision rule by restricting the decision variable as an affine function of the uncertain demand. By reformulating the problem as an *affinely adjustable robust counterpart* (AARC), the problem could then be solved efficiently. With the development of AARC, plenty of applications have been implemented on the various fields, including transmission network, energy, warehouse, supply chain contract, humanitarian logistics areas, and microgrid management [46]–[51]. We refer readers who want to see more detailed examples of the adjustable robust optimization approach to the survey paper written by [52].

Distributionally robust optimization began with [53]. By considering the worst-case distribution in the newsvendor problem, Scarf's ordering rule was established. After that, this approach was facilitated by [54], and later became the cornerstone of the distributionally robust optimization approach. As a result, the utilization of a distributionally robust bound, such as an approximation from the stochastic program, also has been conducted [41]. Chen and Sim [55] utilized the distributionally robust bound to approximate the bound of the expectation of the positive part. By utilizing the distributionally robust bound proposed by Chen and Sim [55] and See and Sim [56] approximated the multiperiod stochastic inventory model to the deterministic second-order cone program. Based on this inventory model, various application studies of the inventory model have been conducted [49], [57], [58].

Recently, the tremendous number of studies that consider omnichannel retailing have been conducted. However, not much has been studied from the perspective of the multiperiod inventory model under an uncertain environment. Although Momen and Torabi [59] adopted the distributionally robust optimization approach in the omnichannel system, they considered the problem as a structure of game theory which is based on the single-period problem. In [60], the robust multiperiod model considering the omnichannel is considered, and the main omnichannel service is ship-from-store service. We summarized the characteristics of our research in Table I, to highlight the distinguishing factors from previous research in the context of supply chain management that looks at the omnichannel system.

III. PROBLEM DESCRIPTION

In this article, we consider a multiperiod inventory model with a periodic review system on a discrete-time planning horizon. The inventory manager of the retailer operates two types of inventories simultaneously, which are online and offline stores. In the online store, two types of customers are considered. One of them is a typical online customer who purchases the product by accessing the online store and then receiving it through the parcel service. The other one is the customer who uses the BOPIS approach. In this case, the customer buys the product through the online store and picks it up by visiting the B&M store personally. The customer who visits the offline store to buy the product is the traditional B&M customer. Throughout this article, the terms B&M demand, pickup demand, and *online demand* will be used to refer to the three types of demands. The BOPIS is the service for the customer who purchases the product through the online store and picks it up at the B&M store (we will use the "offline store" and "B&M store" interchangeably). The OSHDS refers to the home delivery service of products with unsatisfied demand (i.e., products that were out of stock) in the B&M store. That is, the OSHDS is a service that delivers the product to the B&M customer directly from an inventory of an online store when the B&M customer faces stockout in the B&M store. In other words, it is a system for responding to B&M customers, not a specific demand that is separately considered. We narrow the focus in this article to the service that customers receive after visiting an offline store. Some retailers offer BOPIS through a consignment location for distribution warehouses or pickup locations, but the service is limited in this article to the store. We made the following assumptions before developing the mathematical model.

Assumption 1: There is a priority of fulfilment among the three types of demands: BOPIS $>$ B&M store $>$ online store.

The retailer operates the inventory by assigning priority to the three types of demands. Customers who demand the BOPIS option have already paid for the product and visit an offline store to receive it. That is, the BOPIS option is available only when the desired product is in-stock in a B&M store [61]. Accordingly, it is natural that the fulfillment for the BOPIS is determined as the first priority. If the product is out of stock, it may severely impact the retailer's reputation and lead to customer loss. Customers who demand the B&M option but face the out-of-stock situation do not present a problem because the backlog is available to them through the OSHDS option as well. Finally, we assumed the lowest priority for customers who demand the online option.

Assumption 2: The retailer can delay the delivery for the online demand customer.

In practice, many online stores ask for forgiveness of delays by offering free shipping. In addition, customers who cannot wait for a late delivery can receive their item directly by using the BOPIS option. We, therefore, assume that general online store customers are generous with forgiving delivery delays. Given the results of [62], this assumption is fully possible.

Assumption 3: When the retailer delays the order for the customer of the online store, the delay is allowed until the maximum period τ.

Although customers of online stores are assumed to forgive delivery delays, the maximum period for the delay is assumed. We consider the maximum period, τ, to incorporate the delay of the delivery.

Assumption 4: When the retailer places an order from the manufacturer, the lead time L_m is required. Meanwhile, when the retailer sends the express delivery for the OSHDS, the lead time L_r is required.

When a retailer places an order to a manufacturer, it is assumed that a lead time L_m, including production time and delivery time, is required. In the case of the OSHDS, it is assumed that a lead time L_r, which is shorter than L_m, occurs when the delivery is made directly from the online store inventory.

Assumption 5: OSHDS service is fulfilled at the start of each period and sent at the end of each period.

We assume that OSHDS is sent at the end of each period because order decisions are made after observing realized demand at each period. Since we develop the model based on the discrete time planning horizon, demand is assumed to be realized at the end of each period. Accordingly, making a decision of OSHDS at the end of each period is natural with the passage of time.

A. Sequences of Decisions by the Inventory Manager

The inventory manager determines the order quantity to respond to future demand over the period. The inventory manager considers three types of demands, $\tilde{d}_t, \tilde{\phi}_t$, and $\tilde{\xi}_t$, which are B&M, BOPIS, and online demands, respectively, at a period $t \in \mathfrak{T}$. We assume that the inventory manager determines the order quantity after observing the inventory level at the beginning of each period. Although demands will be realized at a given period at any time, we assume that they are realized at the end of each period. It is also assumed that products had been ordered before the lead time L_m and arrived at the end of each period.

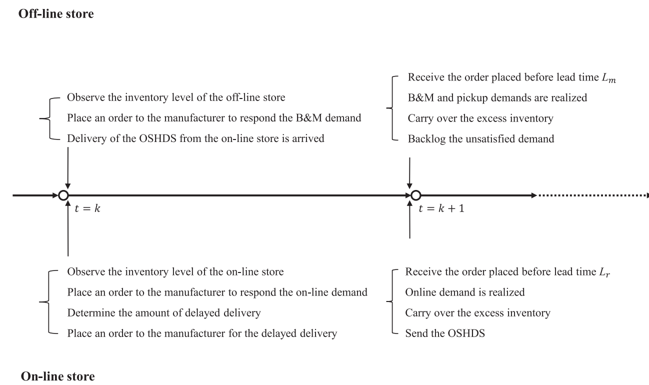

Fig. 3. Decision sequence of the retailer.

TABLE II
PARAMETERS

c_t	Ordering cost per unit from a manufacturer at period t.
$h_{u,t}$	Inventory holding cost per unit of the store at period t.
$h_{v,t}$	Inventory holding cost per unit relevant to the pickup demand at period t.
$h_{w,t}$	Inventory holding cost per unit of the online store at period t.
$b_{u,t}$	Penalty cost per unit of the unsatisfied B&M demand at period t.
$b_{v,t}$	Penalty cost per unit of the unsatisfied pickup demand at period t.
s_t	OSHDS cost per unit at period t.
p_t	Delayed delivery cost per unit at period t.
C_t	Order capacity of the manufacturer at period t.
U_t	Inventory capacity of the B&M store at period t.
K_t	Inventory capacity of the online store at period t.

Products that are not sold at the end of the period are carried over to the next period, whereas unsatisfied demands are assumed to be backlogged to the next period. The sequences of decisions by the retailer are shown in Fig. 3.

IV. Mathematical Formulation

We propose a multiperiod inventory model that incorporates the BOPIS and OSHDS simultaneously. We used parameters and decision variables, which are represented in Tables II and III, respectively. Based on the decision variables and the three types of demands, we made three balance equations, which are shown in Fig. 4. In the case of m_t, it does not physically affect the inventory level. Accordingly, we expressed it in parentheses. For an online demand, the retailer can delay the delivery. On the other hand, if a pickup demand customer visits the B&M store to receive his or her product and the product is out of stock, this can seriously damage the retailer's image. Therefore, the retailer should pay more attention to pickup demand and prioritize the realization of that demand. Hence, we conceptually separate the inventory of the offline store into two categories. Prioritizing the demand of the customer who picks up items in the store can be achieved by dividing the inventory into two categories and imposing an enormous cost penalty to the unsatisfied pickup demand.

TABLE III
DECISION VARIABLES

x_t	Order quantity from a manufacturer to respond to the B&M demand at period t.
y_t	Order quantity from a manufacturer to respond to the pickup demand at period t.
o_t	Order quantity from a manufacturer to respond to the online demand at period t.
q_t	OSHDS quantity from the online store at period t.
m_t	Delayed order quantity to the demand of the online store process at period t.
$m_{t,k}$	Amount to be sent at period k which is delayed at period t.
u_t	Inventory level related to the B&M store at period t.
v_t	Inventory level related to the pickup demand at period t.
w_t	Inventory level related to the online store at period t.

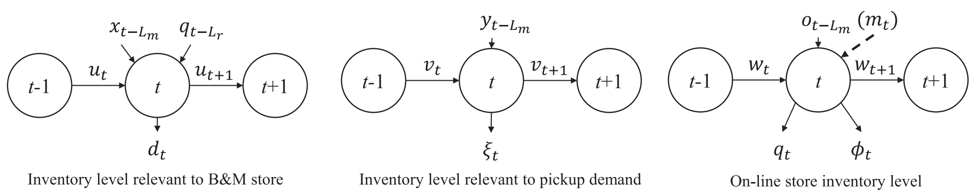

Fig. 4. Three types of balance equations.

TABLE IV
TOTAL COSTS CONSIDERED IN THE OBJECTIVE FUNCTION

Total ordering costs (TOC) $= \sum_{t\in\mathfrak{T}} [c_t (x_t + y_t + o_t)]$
Total OSHDS costs (TIC) $= \sum_{t\in\mathfrak{T}} s_t q_t$
Total delayed delivery costs (TDC) $= \sum_{t\in\mathfrak{T}} p_t m_t$
Total inventory holding costs (THC) $= \sum_{t\in\mathfrak{T}} \left[h_{u,t}(u_{t+1})^+ + h_{v,t}(v_{t+1})^+ + h_{w,t}(w_{t+1})^+\right]$
Total backlog costs (TBC) $= \sum_{t\in\mathfrak{T}} \left[b_{u,t}(u_{t+1})^- + b_{v,t}(v_{t+1})^-\right]$

A. Deterministic Formulation

We begin with the mathematical formulation under the deterministic demand. In this model, all demands throughout the entire planning horizon are regarded as deterministic values. We used notations d_t, ξ_t, and ϕ_t to distinguish the deterministic demands from the uncertain demands, $\tilde{d}_t$, $\tilde{\xi}_t$, and $\tilde{\phi}_t$. The objective function is to minimize the total costs within the entire planning horizon. We considered the following costs in the objective function, as given in Table IV.

We developed the deterministic formulation as follows:

$$\min \quad \text{TOC} + \text{TIC} + \text{TDC} + \text{THC} + \text{TBC} \tag{1}$$

$$\text{s.t.} \quad u_{t+1} = u_t + x_{t-L_m} - d_t + q_{t-L_r} \qquad t \in \mathfrak{T} \tag{2}$$

$$v_{t+1} = v_t + y_{t-L_m} - \xi_t \qquad t \in \mathfrak{T} \tag{3}$$

$$w_{t+1} = w_t + o_{t-L_m} + m_t - q_t - \phi_t \qquad t \in \mathfrak{T} \tag{4}$$

$$m_t = \sum_{k=1}^{\tau} m_{t,t-1+k} \qquad t \in \mathfrak{T} \tag{5}$$

$$(u_t + x_{t-L_m} - d_t)^- \geq q_{t-L_r} \qquad t \in \mathfrak{T} \tag{6}$$

$$(w_t + o_{t-L_m} - q_t - \phi_t)^- \geq m_t \qquad t \in \mathfrak{T} \tag{7}$$

$$x_t + y_t + o_t + \sum_{k=1}^{\tau} m_{t+1-k,t} \leq C_t \qquad t \in \mathfrak{T} \tag{8}$$

$$u_t + v_t \leq U_t \qquad t \in \mathfrak{T}^+ \tag{9}$$

$$w_t + o_{t-L_m} - q_t - \phi_t \leq K_t \qquad t \in \mathfrak{T} \tag{10}$$

$$x_t, y_t, o_t, m_t, \text{ and } q_t \geq 0 \qquad t \in \mathfrak{T}. \tag{11}$$

The objective function (1) minimizes the total costs, within the entire planning horizon. Constraints (2) represent the balance equation among the order quantity, B&M demand, inventory level, and realized OSHDS. Constraints (3) represent the balance equation among the pickup demand, relative inventory level, and order quantity. Constraints (4) represent the balance equation among the online demand, relevant inventory level, order quantity, sent OSHDS, and express delivery for the online demand. Constraints (5) indicate the delivery to be completed during an acceptable delay period τ. Constraints (6) and (7) are the logical expressions that OSHDS and delayed delivery are operated when the relevant inventories have negative numbers. Constraints (8) indicates the order capacity of the manufacturer. Constraints (9) and (10) represent the inventory capacities of the offline and online stores, respectively. Nonnegative conditions of the real variables are presented in Constraints (11). According to Constraints (2) and (6), the OSHDS quantity from an online store is transported when the inventory level of the B&M store in Constraints (2) becomes negative number. In a manner similar to Constraints (4) and (7), delayed delivery is transported when the inventory level of an online store becomes negative number.

In the case of the Constraints (6), the left-hand side $(\cdot)^-$ are reformulated to linear inequalities by introducing new auxiliary variables, e_t^- and e_t^+, as follows:

$$\begin{aligned}&(u_t + x_{t-L_m} - d_t)^- \geq q_{t-L_r}\\ &\iff \max(0, -u_t - x_{t-L_m} + d_t) \geq q_{t-L_r}\\ &\iff \begin{cases} q_{t-L_r} \leq e_t^- + e_t^+ \\ q_{t-L_r} \leq u_t - x_{t-L_m} + d_t + e_t^- + e_t^+ \\ e_t^- \geq -u_t - x_{t-L_m} + d_t \\ e_t^+ \geq u_t + x_{t-L_m} - d_t \\ \text{where } e_t^-, e_t^+ \geq 0. \end{cases}\end{aligned} \tag{12}$$

In a similar manner, we reformulated Constraints (7) as follows:

$$\begin{aligned}&(w_t + o_{t-L_m} - q_t - \phi_t)^- \geq m_t\\ &\iff \max(0, -w_t - o_{t-L_m} + q_t + \phi_t) \geq m_t\\ &\iff \begin{cases} m_t \leq r_t^- + r_t^+ \\ m_t \leq w_t - o_{t-L_m} + q_t + \phi_t + r_t^- + r_t^+ \\ r_t^- \geq -w_t - o_{t-L_m} + q_t + \phi_t \\ r_t^+ \geq w_t + o_{t-L_m} - q_t - \phi_t \\ \text{where } r_t^-, r_t^+ \geq 0. \end{cases}\end{aligned} \tag{13}$$

In this article, we developed a mathematical formulation as a linear program. In practice, a fixed ordering cost or integer variable could be considered. However, to utilize the distributionally robust optimization approach, we developed the mathematical formulation as a linear program as relevant previous studies did [49], [56].

B. *Factor-Based Demand Model*

In the omnichannel system, the interaction between each channel should be considered [63]. From a perspective of the demand modeling, correlations among uncertain demands should be regarded. In practice, precisely estimating the probability distribution from past data is difficult. To capture the correlations among uncertain demands based on partial information from the descriptive statistics on data, we adopted a factor-based demand model for $t \in \mathfrak{T}$ as follows:

$$\tilde{d}_t(\tilde{\mathbf{z}}_t) \triangleq d_t^0 + \sum_{k=1}^{N} d_t^k \tilde{z}_k$$

$$\tilde{\phi}_t(\tilde{\mathbf{z}}_t) \triangleq \phi_t^0 + \sum_{k=1}^{N} \phi_t^k \tilde{z}_k$$

$$\tilde{\xi}_t(\tilde{\mathbf{z}}_t) \triangleq \xi_t^0 + \sum_{k=1}^{N} \xi_t^k \tilde{z}_k$$

$1 \leq N_1 \leq N_2 \leq \cdots \leq N_{T-1} = N$ and $\tilde{\mathbf{z}}_k$ are unfolded until $k = 1, \ldots, N_t$.

The notations d_t^0, ϕ_t^0, and ξ_t^0 represent the nominal values of B&M, pickup, and online demands, respectively. Perturbations of demands are represented by the primitive uncertain factors $\tilde{z}_k$, and estimated coefficients d_t^k, ϕ_t^k, and ξ_t^k. By developing the demand model as affinely dependent on the uncertain factors, which means that at least one of the uncertain factors can be defined as an affine combination of the others, correlations among demands, or crossover during the period can be characterized. We made assumptions of uncertain factors $\tilde{\mathbf{z}}$ in Assumption A.

Assumption A: Uncertain factors $\tilde{\mathbf{z}}$ are random variables whose first and second moments are zero mean and positive definite covariance matrix $\mathbf{\Sigma}$, respectively. Although the distribution is unknown, each factor is distributed on the second-order conic representable set $\boldsymbol{W}$, including interval, polyhedron, and ellipsoid sets.

As shown in Assumption A, we only consider partial information about distribution which characterizes an ambiguity demand. If the demand model is regarded as a random variable with the full information about the probability distribution, the mathematical model becomes a multistage stochastic program, which is represented as follows:

$$\begin{aligned}
\min\quad & \text{STOC} + \text{STIC} + \text{STDC} + \text{STHC} + \text{STBC}\\
\text{s.t.}\quad & u_{t+1}(\tilde{\mathbf{z}}_t) = u_t(\tilde{\mathbf{z}}_{t-1}) + x_{t-L_m}(\tilde{\mathbf{z}}_{t-L_m-1})\\
&\qquad - d_t(\tilde{\mathbf{z}}_t) + q_{t-L_r}(\tilde{\mathbf{z}}_{t-L_r-1}) \qquad t \in \mathfrak{T}\\
& v_{t+1}(\tilde{\mathbf{z}}_t) = v_t(\tilde{\mathbf{z}}_{t-1}) + y_{t-L_m}(\tilde{\mathbf{z}}_{t-L_m-1})\\
&\qquad - \xi_t(\tilde{\mathbf{z}}_t) \qquad t \in \mathfrak{T}\\
& w_{t+1}(\tilde{\mathbf{z}}_t) = w_t(\tilde{\mathbf{z}}_{t-1}) + o_{t-L_m}(\tilde{\mathbf{z}}_{t-L_m-1})\\
&\qquad + m_t(\tilde{\mathbf{z}}_{t-1}) - q_t(\tilde{\mathbf{z}}_{t-1}) - \phi_t(\tilde{\mathbf{z}}_t) \qquad t \in \mathfrak{T}\\
& m_t(\tilde{\mathbf{z}}_{t-1}) = \sum_{k=1}^{\tau} m_{t,t-1+k}(\tilde{\mathbf{z}}_{t-1}) \qquad t \in \mathfrak{T}\\
& (u_t(\tilde{\mathbf{z}}_{t-1}) + x_{t-L_m}(\tilde{\mathbf{z}}_{t-L_m-1}) - d_t(\tilde{\mathbf{z}}_t))^-\\
& \geq q_{t-L_r}(\tilde{\mathbf{z}}_{t-L_r-1}) \qquad t \in \mathfrak{T}\\
& (w_t(\tilde{\mathbf{z}}_{t-1}) + o_{t-L_m}(\tilde{\mathbf{z}}_{t-L_m-1}) - q_t(\tilde{\mathbf{z}}_{t-1})\\
&\qquad - \phi_t(\tilde{\mathbf{z}}_t))^- \geq m_t(\tilde{\mathbf{z}}_{t-1}) \qquad t \in \mathfrak{T}\\
& x_t(\tilde{\mathbf{z}}_{t-1}) + y_t(\tilde{\mathbf{z}}_{t-1}) + o_t(\tilde{\mathbf{z}}_{t-1})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\tau} m_{t+1-k,t}(\tilde{\mathbf{z}}_{t-1}) \le C_t && t\in\mathfrak{T}\\
&u_t(\tilde{\mathbf{z}}_{t-1})+v_t(\tilde{\mathbf{z}}_{t-1}) \le U_t && t\in\mathfrak{T}^+\\
&w_t(\tilde{\mathbf{z}}_{t-1})+o_{t-L_m}(\tilde{\mathbf{z}}_{t-L_m-1})-q_t(\tilde{\mathbf{z}}_{t-1})\\
&\qquad -\phi_t(\tilde{\mathbf{z}}_t)\le K_t && t\in\mathfrak{T}\\
&x_t(\tilde{\mathbf{z}}_{t-1}), y_t(\tilde{\mathbf{z}}_{t-1}), o_t(\tilde{\mathbf{z}}_{t-1}), m_t(\tilde{\mathbf{z}}_{t-1}),\\
&\qquad \text{and } q_t(\tilde{\mathbf{z}}_{t-1})\ge 0 && t\in\mathfrak{T}
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
\text{STOC} &= \mathbb{E}\left[\sum_{t\in\mathfrak{T}} c_t\left(x_t(\tilde{\mathbf{z}}_{t-1})+y_t(\tilde{\mathbf{z}}_{t-1})+o_t(\tilde{\mathbf{z}}_{t-1})\right)\right]\\
\text{STIC} &= \mathbb{E}\left[\sum_{t\in\mathfrak{T}}\left(s_t q_t(\tilde{\mathbf{z}}_{t-1})\right)\right]\\
\text{STDC} &= \mathbb{E}\left[\sum_{t\in\mathfrak{T}}\left(p_t m_t(\tilde{\mathbf{z}}_{t-1})\right)\right]\\
\text{STHC} &= \mathbb{E}\Bigg[\sum_{t\in\mathfrak{T}}\big[h_{u,t}(u_{t+1}(\tilde{\mathbf{z}}_t))^+ + h_{v,t}(v_{t+1}(\tilde{\mathbf{z}}_t))^+\\
&\qquad + h_{w,t}(w_{t+1}(\tilde{\mathbf{z}}_t))^+\big)\Bigg]\\
\text{STBC} &= \mathbb{E}\left[\sum_{t\in\mathfrak{T}}\left(b_{u,t}(u_{t+1}(\tilde{\mathbf{z}}_t))^- + b_{v,t}(v_{t+1}(\tilde{\mathbf{z}}_t))^-\right)\right].
\end{aligned}
\tag{15}
$$

Based on the decision sequences, which are illustrated in Fig. 3, we restrict the unfolded uncertain factors in the stochastic program (14).

1) Linear Order Policy (LOP): Unfortunately, evaluating the expectation of the multistage stochastic program is computationally intractable. To overcome the limitation of the stochastic program, we utilized the rule-based approximation. By restricting the decision variable as an affine function of the uncertain factors $\tilde{\mathbf{z}}$, the multistage decision process under the uncertain demand could be derived to the tractable formulation [46], [56]. Throughout the rest of this article, we will use the term LOP to refer to the linearly restricted decision variable

$$
\begin{aligned}
x_t(\tilde{\boldsymbol{z}}) &= x_t^0+\sum_{k=1}^{N} x_t^k \tilde{z}_k \qquad y_t(\tilde{\boldsymbol{z}}) = y_t^0+\sum_{k=1}^{N} y_t^k \tilde{z}_k\\
o_t(\tilde{\boldsymbol{z}}) &= o_t^0+\sum_{k=1}^{N} o_t^k \tilde{z}_k \qquad q_t(\tilde{\boldsymbol{z}}) = q_t^0+\sum_{k=1}^{N} q_t^k \tilde{z}_k\\
m_t(\tilde{\boldsymbol{z}}) &= m_t^0+\sum_{k=1}^{N} m_t^k \tilde{z}_k.
\end{aligned}
\tag{16}
$$

To make the decision based on the realized uncertain factors, which is referred to as the *nonanticipative* property, uncertain factors, which are not observable in each period, could be restricted. By adding the following constraints, this property can be incorporated:

$$
x_t^k, y_t^k, o_t^k, q_t^k, m_t^k = 0 \qquad \forall k \ge N_{t-1}+1. \tag{17}
$$

Hence, the retailer could establish the order decision based on the realized uncertain factors at the beginning of each period t.

Remark 1: The decision variable $m_{t,t-1+k}$ is also linearly dependent on uncertain factors, $\tilde{z}_k$, as follows:

$$
\begin{aligned}
m_t(\tilde{\mathbf{z}}_{t-1}) &= \sum_{i=1}^{\tau} m_{t,t-1+i}(\tilde{\mathbf{z}}_{t-1})\\
\iff m_t^0+\sum_{k=1}^{N} m_t^k\tilde{z}_k &= m_{t,t}^0+m_{t,t+1}^0+\cdots+m_{t,t-1+\tau}^0\\
&\quad+\sum_{k=1}^{N}\left(m_{t,t}^k+m_{t,t+1}^k+\cdots+m_{t,t-1+\tau}^k\right)\tilde{z}_k\\
&= \sum_{i=1}^{\tau}\left(m_{t,t-1+i}^0+\sum_{k=1}^{N}\left(m_{t,t-1+i}^k\tilde{z}_k\right)\right).
\end{aligned}
$$

Remark 2: The decision variables of inventory level u_{t+1} are affine functions of the uncertain factors as follows:

$$
\begin{aligned}
u_{t+1}(\tilde{z}) &= u_1^0+\sum_{i=1}^{t} x_{i-L_m}(\tilde{z})-\sum_{i=1}^{t} d_i(\tilde{z})+\sum_{i=1}^{t} q_{i-L_r}(\tilde{z})\\
&= u_1^0+\sum_{i=1}^{t}((x_{i-L_m}^0-d_i^0+q_{i-L_r}^0)\\
&\quad+\sum_{k=1}^{N}(x_{i-L_m}^k-d_i^k+q_{i-L_r}^k)\tilde{z}_k)\\
&= u_t^0+\sum_{k=1}^{N} u_t^k\tilde{z}_k.
\end{aligned}
\tag{18}
$$

In the same manner, inventory levels v_{t+1} and w_{t+1} also feature linear functions of the uncertain factors. The LOP formulation of the inventory level $u_{t+1}(\tilde{z})$ also features the nonanticipative property. In the same manner, decision variables $v_t(\tilde{z})$ and $w_t(\tilde{z})$ become linear functions of the predefined uncertain factors and feature nonanticipative properties.

Remark 3: The auxiliary variables, e_t^-, e_t^+, r_t^-, and r_t^+, represented in (12) and (13), are also affected by the uncertain factors in the stochastic formulation (14). Accordingly, the auxiliary variables are formulated as linearly dependent on the uncertain factors as follows:

$$
\begin{aligned}
e_t^-(\tilde{z}) &= e_{t,0}^-+\sum_{k=1}^{N} e_{t,k}^-\tilde{\boldsymbol{z}}_k \qquad e_t^+(\tilde{z}) = e_{t,0}^++\sum_{k=1}^{N} e_{t,k}^+\tilde{\boldsymbol{z}}_k\\
r_t^-(\tilde{z}) &= r_{t,0}^-+\sum_{k=1}^{N} r_{t,k}^-\tilde{\boldsymbol{z}}_k \qquad r_t^+(\tilde{z}) = r_{t,0}^++\sum_{k=1}^{N} r_{t,k}^+\tilde{\boldsymbol{z}}_k.
\end{aligned}
$$

2) Upper Bound of $\mathbb{E}[(\cdot)^+]$: The objective function of the multistage stochastic program consists of ordering cost (STOC), OSHDS cost (STIC), delivery cost (STDC), inventory holding cost (STHC), and backlog cost (STBC). When the LOP is introduced to the multistage stochastic program, STOC, STIC, and STDC can easily be handled under the Assumption A as follows:

$$\mathbb{E}\left(c_t x_t(\tilde{z}_{t-1})\right) = \mathbb{E}\left(c_t(x_t^0 + \sum_{k=1}^{N} x_t^k \tilde{z}_k)\right) = c_t x_t^0. \quad (19)$$

In the same way, as shown in (20), both STIC and STDC can be reformulated by using LOP. Accordingly, STOC, STIC, and STDC, are reformulated as RTOC, RTIC, and RTDC as follows:

$$\begin{aligned} \text{RTOC} &= \sum_{t\in\mathfrak{T}} c_t x_t^0 \\ \text{RTIC} &= \sum_{t\in\mathfrak{T}} s_t q_t^0 \\ \text{RTDC} &= \sum_{t\in\mathfrak{T}} p_t m_t^0. \end{aligned} \quad (20)$$

Due to the expected positive and negative parts in the objective function (15), STHC and STBC could not be derived directly as shown in (19). To handle these cumbersome parts, we utilized the distributionally robust bound proposed by [55]. A close upper bound of the expected positive part could be obtained by utilizing the LOP under the ambiguity uncertain demand. By adopting the work of [55], three types of upper bound functions of $\mathbb{E}((y_0 + \mathbf{y}'\tilde{\mathbf{z}})^+)$ can be defined as follows:

$$\begin{aligned} \pi_1(y_0, \mathbf{y}) &\triangleq \left(y_0 + \max_{\tilde{z}\in\mathcal{W}} \tilde{\mathbf{z}}'\mathbf{y}\right)^+ \\ \pi_2(y_0, \mathbf{y}) &\triangleq y_0 + \left(-y_0 + \max_{\tilde{z}\in\mathcal{W}} \tilde{\mathbf{z}}'(-\mathbf{y})\right)^+ \\ \pi_3(y_0, \mathbf{y}) &\triangleq \frac{1}{2}y_0 + \frac{1}{2}\left|y_0, \Sigma^{1/2}\mathbf{y}_3\right|_2. \end{aligned}$$

Theorem 1: By minimizing the three upper bounds, π_1, π_2, and π_3, the tighter upper bound π can be derived [55]

$$\begin{aligned} \pi(y_0, \mathbf{y}) \triangleq \min_{y_i^0, \mathbf{y_i}} \quad & \pi_1(y_i^0, \mathbf{y_i}) + \pi_2(y_i^0, \mathbf{y_i}) + \pi_3(y_i^0, \mathbf{y_i}) \\ \text{s.t.} \quad & y_1^0 + y_2^0 + y_2^0 = y_0 \\ & \mathbf{y_1} + \mathbf{y_2} + \mathbf{y_3} = \mathbf{y}. \end{aligned}$$

Remark 4: We utilized only three upper bounds, unlike Chen and Sim [55], who proposed five types of bounds.

See and Sim [56] mentioned the bound obtained by minimizing three bounds could provide a sufficiently tight bound. The remaining two bounds, which are not covered in this article, could be interpreted as the infinite values of the forward and backward deviations, which are acceptable characteristics in practice.

By utilizing the distributionally robust bound, the tighter upper bound, $\pi(y_0, \mathbf{y})$, could be obtained with the following optimization problem, which is represented as epigraph form:

$$\begin{aligned} \pi(y^0, \mathbf{y}) = \min \quad & M_1 + M_2 + M_3 \\ \text{s.t.} \quad & y_1^0 + \max_{\tilde{\mathbf{z}}\in\mathbf{W}} \tilde{\mathbf{z}}'\mathbf{y}_1 \le M_1 \\ & M_1 \ge 0 \\ & \max_{\tilde{\mathbf{z}}\in\mathbf{W}} \tilde{\mathbf{z}}'(-\mathbf{y}_2) \le M_2 \\ & y_2^0 \le M_2 \\ & \frac{1}{2}y_3^0 + \frac{1}{2}\left|y_3^0, \Sigma^{1/2}\mathbf{y}_3\right|_2 \le M_3 \\ & y_1^0 + y_2^0 + y_3^0 = y_0 \\ & \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 = \mathbf{y} \\ & M_i, y_i^0 \in \mathbb{R},\ \mathbf{y}_i \in \mathbb{R}^N,\ i = 1, \ldots, 3. \end{aligned} \quad (21)$$

Theorem 2: By solving the optimization problem (21), the tighter bound can be achieved as the following inequality presents [55]:

$$\mathbb{E}\left((y_0 + \mathbf{y}'\tilde{\mathbf{z}})^+\right) \le \pi(y^0, \mathbf{y}) \le \min_{i=1,2,3} \pi_i(y_i^0, \mathbf{y}_i). \quad (22)$$

Hence, we could derive the upper bound of the STHC, which is represented as, RTHC, as shown in the following:

$$\begin{aligned} \text{RTHC} = \sum_{t\in\mathfrak{T}} \big(& h_{u,t}\pi(u_{t+1}^0, \mathbf{u}_{t+1}) + h_{v,t}\pi(v_{t+1}^0, \mathbf{v}_{t+1}) \\ & + h_{w,t}\pi(w_{t+1}^0, \mathbf{w}_{t+1}) \big). \end{aligned} \quad (23)$$

Theorem 3 ([55]): In the case of the upper bound of $\mathbb{E}(\cdot)^-$, it could be derived as follows:

$$\mathbb{E}\left((y_0 + \mathbf{y}'\tilde{\mathbf{z}})^-\right) \le \pi(-y^0, -\mathbf{y}) \le \min_{i=1,2,3} \pi_i(-y_i^0, -\mathbf{y}_i). \quad (24)$$

Accordingly, we could derive the RTBC from the STBC, as shown in the following:

$$\text{RTBC} = \sum_{t\in\mathfrak{T}} \left(b_{u,t}\pi(-u_{t+1}^0, -\mathbf{u}_{t+1}) + b_{v,t}\pi(-v_{t+1}^0, -\mathbf{v}_{t+1}) \right). \quad (25)$$

3) Affinely Adjustable Robust Counterpart: Based on the approximated upper bound from (15), which consists of (20), (23), and (25), an AARC of the LOP formulation is derived as follows:

$$\min \quad \text{RTOC} + \text{RTIC} + \text{RTDC} + \text{RTHC} + \text{RTBC} \quad (26)$$

$$\text{s.t.} \quad \begin{cases} u_{t+1}^k = u_t^k + x_{t-L_m}^k - d_t^k + q_{t-L_r}^k \\ v_{t+1}^k = v_t^k + y_{t-L_o}^k + m_{t-L_m}^k - \xi_t^k \\ w_{t+1}^k = w_t^k + o_{t-L_m}^k + m_t^k - q_t^k - \phi_t^k \end{cases} \quad (27)$$

$$k \in 0 \ldots N; t \in \mathfrak{T}$$

$$\begin{cases} x_t^k,\ y_t^k,\ o_t^k,\ q_t^k,\ m_t^k = 0 & k \ge N_{t-1} + 1;\ t \in \mathfrak{T}; \\ u_{t+1}^k,\ v_{t+1}^k,\ w_{t+1}^k = 0 & k \ge N_t + 1;\ t \in \mathfrak{T}; \end{cases} \quad (28)$$

$$\left\{ m_t^k = \sum_{i=1}^{\tau} m_{t,t-1+i}^k \quad k \in 0 \ldots N; t \in \mathfrak{T}; \right. \quad (29)$$

$$\begin{cases} q_{t-L_r}^k \le e_{t,k}^- + e_{t,k}^+ \\ q_{t-L_r}^k \le u_t^k - x_{t-L_m}^k + d_t^k + e_{t,k}^- + e_{t,k}^+ \\ e_{t,k}^- \ge -u_t^k - x_{t-L_m}^k + d_t^k \\ e_{t,k}^+ \ge u_t^k + x_{t-L_m}^k - d_t^k \end{cases} \quad (30)$$

$$k \in 0 \ldots N; t \in \mathfrak{T}$$

$$\begin{cases} m_t^k \le r_{t,k}^- + r_{t,k}^+ \\ m_t^k \le w_t^k - o_{t-L_m}^k + \phi_t^k + r_{t,k}^- + r_{t,k}^+ \\ r_{t,k}^- \ge -w_t^k - o_{t-L_m}^k + \phi_t^k \\ r_{t,k}^+ \ge w_t^k + o_{t-L_m}^k - \phi_t^k \end{cases} \tag{31}$$

$$k \in 0 \dots N; t \in \mathfrak{T}$$

$$\begin{cases} x_t^0 + y_t^0 + o_t^0 + \sum_{i=1}^{\tau} m_{t+1-i,t}^0 + \sum_{k=1}^{N} \\ \left(x_t^k + y_t^k + o_t^k + \sum_{i=1}^{\tau} m_{t+1-i,t}^k\right) \tilde{z}_k \le C_t \\ w_t^0 + o_{t-L_m}^0 - q_t^0 - \phi_t^0 + \sum_{k=1}^{N} \\ \left(w_t^k + o_{t-L_m}^k - q_t^k - \phi_t^k\right) \tilde{z}_k \le K_t \end{cases} \tag{32}$$

$$\forall k, \tilde{z}_k \in \boldsymbol{W};\ t \in \mathfrak{T}$$

$$\left\{ u_t^0 + v_t^0 + \sum_{k=1}^{N} \left(u_t^k + v_t^k\right) \tilde{z}_k \le U_t \right. \tag{33}$$

$$\forall k, \tilde{z}_k \in \boldsymbol{W};\ t \in \mathfrak{T}^+$$

$$\begin{cases} x_t^0 + \sum_{k=1}^{N} x_t^k \tilde{z}_k \ge 0,\ y_t^0 + \sum_{k=1}^{N} y_t^k \tilde{z}_k \ge 0 \\ q_t^0 + \sum_{k=1}^{N} q_t^k \tilde{z}_k \ge 0,\ m_t^0 + \sum_{k=1}^{N} m_t^k \tilde{z}_k \ge 0 \\ e_{t,0}^- + \sum_{k=1}^{N} e_{t,k}^- \tilde{z}_k \ge 0,\ e_{t,0}^+ + \sum_{k=1}^{N} e_{t,k}^+ \tilde{z}_k \ge 0 \\ r_{t,0}^- + \sum_{k=1}^{N} r_{t,k}^- \tilde{z}_k \ge 0,\ r_{t,0}^+ + \sum_{k=1}^{N} r_{t,k}^+ \tilde{z}_k \ge 0 \end{cases}$$

$$\forall k, \tilde{z}_k \in \boldsymbol{W};\ t \in \mathfrak{T}. \tag{34}$$

Constraints (27)–(31) feature linear forms in the previous mathematical model. However, uncertain factors still remain in Constraints (32) and (33). To establish these constraints as the robust counterpart, we defined the inner optimization problem in each constraint as follows:

$$x_t^0 + y_t^0 + o_t^0 + \sum_{i=1}^{\tau} m_{t+1-i,t}^0 + \sum_{k=1}^{N} \left(x_t^k + y_t^k + o_t^k + \sum_{i=1}^{\tau} m_{t+1-i,t}^k\right) \tilde{z}_k \le C_t, \quad \tilde{z}_k \in \boldsymbol{W}$$

$$\iff x_t^0 + y_t^0 + o_t^0 + \sum_{i=1}^{\tau} m_{t+1-i,t}^0 + \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^{N} \left(x_t^k + y_t^k + o_t^k + \sum_{i=1}^{\tau} m_{t+1-i,t}^k\right) \tilde{z}_k \le C_t \tag{35}$$

$$u_t^0 + v_t^0 + \sum_{k=1}^{N} \left(u_t^k + v_t^k\right) \tilde{z}_k \le U_t, \quad \tilde{\mathbf{z}} \in \boldsymbol{W}$$

$$\iff u_t^0 + v_t^0 + \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^{N} (u_t^k + v_t^k) \tilde{z}_k \le U_t \tag{36}$$

$$w_t^0 + o_{t-L_m}^0 - q_t^0 - \phi_t^0 + \sum_{k=1}^{N} \left(w_t^k + o_{t-L_m}^k - q_t^k - \phi_t^k\right) \tilde{z}_k \le K_t, \quad \tilde{\mathbf{z}} \in \boldsymbol{W}$$

$$\iff w_t^0 + o_{t-L_m}^0 - q_t^0 - \phi_t^k + \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^{N} \left(w_t^k + o_{t-L_m}^k - q_t^k - \phi_t^k\right) \tilde{z}_k \le K_t. \tag{37}$$

We also developed robust counterparts for nonnegative conditions of decision variables (34) by defining the inner optimization problem as follows:

$$x_t^0 + \sum_{k=1}^{N} x_t^k \tilde{z}_k \ge 0, \quad \tilde{\mathbf{z}} \in \boldsymbol{W}$$

$$\iff x_t^0 - \max_{\tilde{\mathbf{z}} \in \boldsymbol{W}} \sum_{k=1}^{N} x_t^k \tilde{z}_k \ge 0. \tag{38}$$

In the same manner, as we reformulated the nonnegative condition in (38), other decision variables, $y_t(\tilde{z}_k)$, $q_t(\tilde{z}_k)$, and $m_t(\tilde{z}_k)$, and auxiliary variables, $e_t^-(\tilde{\boldsymbol{z}})$, $e_t^+(\tilde{\boldsymbol{z}})$, $r_t^-(\tilde{\boldsymbol{z}})$, and $r_t^+(\tilde{\boldsymbol{z}})$, are reformulated to the robust counterparts. Under Assumption A, the inner optimization problem could be reformulated to the equivalent second-order cone program (including the linear program) because we assumed that the uncertain factors lie on the second-order conic representable set. The robust counterpart depends on the choice of the uncertain set. When the polyhedron set is chosen, the robust counterpart will feature the linear program [45]. On the other hand, the second-order cone program will be developed when the ellipsoid uncertain set is considered [43]. More details can be found in previous relevant studies [49], [56].

To sum up, we adopted the distributionally robust optimization approach to overcome the limitations of the stochastic optimization model: 1) estimating full information about the demand distribution and 2) tractability issue. Distributionally robust optimization is a methodology that takes into account the worst-case distribution or worst-case expected cost under an ambiguity set, containing a true distribution, rather than assuming a specific distribution for uncertain data. It can be characterized by considering an ambiguity demand through partial information obtained from descriptive statistics, such as means, covariance, and supports. To solve the multistage decision process under the ambiguity demand, an adjustable robust optimization based on wait-and-see decision could be considered. However, an adjustable robust counterpart is generally NP-hard, which is computationally demanding [46]. Accordingly, we utilized a rule-based approximation and a distributionally robust bound to derive the tractable formulation. As a result, the deterministic second-order cone program, which could be solved by the interior point method within polynomial time, is derived. Based on the derived formulation, we conducted computational experiments and reported computational results in Section V.

V. Computational Experiments

A. Demand Process

The factor-based demand model can capture the dependency or correlation of different demands across the periods by estimating the coefficients of the uncertain factors statistically from historical data [49]. The auto-regressive moving average (ARMA) model, which is the representative methodology of predicting future demand as the stochastic process, is modeled from past data. In addition, the vector autoregressive *moving* *average* (VARMA) model can be used to capture correlation or dependency among demands as well as its own past data. If the VARMA or ARMA process is invertible and stable, the process could be represented as a *moving average* (MA) process [64]. See and Sim [56] utilized the demand process by regarding the factor-based demand model as an MA process, which was proposed by Graves [65] in the inventory model. Although the stochastic process regards the uncertain part as a white noise, which follows the normal distribution with a zero mean and an estimated covariance (or variance), it could be truncated to the appropriate support set. That is, the white noise could be induced to the uncertain factor, which means the stochastic process could be regarded as the factor-based demand model. Based on the demand model proposed by [65], the factor-based demand model can be regarded as follows:

$$\begin{aligned}\tilde{d}_t(\tilde{\mathbf{z}}_t) &= \mu + \tilde{z}_t + \alpha\tilde{z}_{t-1} + \alpha\tilde{z}_{t-2} + \cdots + \alpha\tilde{z}_1 \\ &= \mu + \tilde{z}_t + \sum_{k=1}^{t-1} \alpha\tilde{z}_k \\ &= d_{t-1}(\tilde{\mathbf{z}}_{t-1}) - (1-\alpha)\tilde{z}_{t-1} + \tilde{z}_t\end{aligned}$$

where μ and α indicate the mean value and weighted value of the impact of past errors on the present value, respectively.

When $\alpha = 1$, this stochastic process features a random walk, while the demand process is stationary independent and identically distributed when $\alpha = 0$. From 0 to 1, when the α increases, the stochastic process features the property of the nonstationary process [56]. In sum, we estimate a VARMA model to incorporate the dependency of different demands across the different periods, and then induce it as a factor-based demand model.

B. Experiment 1: Verification of the AARC

We used the commercially available software Xpress-MP to conduct computational experiments. First, we conducted experiments to verify the performance of the AARC. Since the derived model provides the objective value as an upper bound of the expected cost, we need to check the loss during the approximation process. We compared the objective value with the *expected value given perfect information* (EV | PI) as a benchmark. As mentioned in Section IV, evaluating the expected value in the multistage decision process is generally intractable. As an alternative to the multistage stochastic model, EV | PI could be utilized [49]. We calculated the EV | PI by generating 10 000 samples of demands and computing the average value of the objective value of the deterministic formulation. Also, we conducted simulation experiments to verify the LOP. The LOP is not the fixed order quantity, but the decision rule over the planning horizon. Thus, the actual order quantity is dependent on the realized uncertain factors. To verify the upper bound of the expected value, we conducted simulation experiments 10 000 times, based on the randomly realized uncertain factors. All parameters used for the computational experiments are generated randomly. (See Table IX in Appendix.) The results of the experiments are tabulated in Table V. As can be seen from Table V, there were no significant differences between the expected upper bound of the LOP formulation and the EV | PI. In other words, the loss in the approximation process was relatively small and, a reasonable order policy was provided. Also, the results of the simulation inventory runs show that the LOP did not show much variability, according to the 10 000 times we ran the simulation experiments. The mean value did not significantly differ from the expected upper bound of the LOP. If more simulations are performed, they will converge to a closer value. We believe the LOP could be utilized even if the demand volatility is high.

TABLE V
RESULTS OF EXPERIMENT 1: VERIFICATION OF THE PERFORMANCE OF AARC

$T = 20$	Expected cost			Simulated inventory runs		
$\lvert\alpha\rvert \leq$	EV|PI	LOP	Gap (%)	Mean	Minimum	Maximum
0	47 930.3	48 607.8	1.41	49 024.4	47 867.6	52 900.9
0.1	47 929.8	48 641.1	1.48	49 041.8	47 903.2	52 689.4
0.2	47 929.0	48 708.5	1.63	49 168.9	47 819.0	52 716.8
0.3	47 931.2	48 724.3	1.65	49 108.2	47 794.4	53 613.5
0.4	47 935.8	48 836.0	1.88	49 281.7	47 614.0	54 036.4
0.5	47 938.5	48 868.1	1.94	49 250.9	47 360.5	54 075.0
$T = 30$	Expected cost			Simulated inventory runs		
$\lvert\alpha\rvert \leq$	EV|PI	LOP	Gap (%)	Mean	Minimum	Maximum
0	70 610.7	71 501.7	1.26	72 376.3	70 728.7	80 300.7
0.1	70 608.4	71 561.1	1.35	72 424.7	70 684.8	80 668.1
0.2	70 611.8	71 614.8	1.42	72 412.9	70 434.8	78 891.8
0.3	70 613.3	71 722.5	1.57	72 490.8	70 394.3	79 448.1
0.4	70 613.1	71 853.0	1.76	72 410.2	70 341.6	78 835.3
0.5	70 613.4	72 171.0	2.21	79 448.1	70 332.4	80 439.8
$T = 40$	Expected cost			Simulated inventory runs		
$\lvert\alpha\rvert \leq$	EV|PI	LOP	Gap (%)	Mean	Minimum	Maximum
0	93061.0	94152.3	1.17	95141.0	93485.0	103032
0.1	93060.5	94235.1	1.26	95213.3	93491.9	103294
0.2	93064.0	94314.4	1.34	95393.7	93237.7	103630
0.3	93063.6	94340.8	1.37	95414.7	93088.3	104828
0.4	93065.5	94611.9	1.66	95863.0	93054.5	108345
0.5	93054.4	95031.0	2.12	96266.1	92668.2	109059
$T = 50$	Expected cost			Simulated inventory runs		
$\lvert\alpha\rvert \leq$	EV|PI	LOP	Gap (%)	Mean	Minimum	Maximum
0	113 783	115 073	1.13	116 835	114 337	129 489
0.1	113 784	115 147	1.20	116 841	114 468	129 301
0.2	113 784	115 227	1.27	116 969	114 324	128 742
0.3	113 790	115 482	1.49	117 671	113 540	135 289
0.4	113 783	116 018	1.96	117 891	114 058	134 416
0.5	113 783	116 912	2.75	118 443	114 262	133 863

C. Experiment 2: Effect of the Support of Uncertain Factors

We performed the computational experiments to examine the effect of the support of the uncertain factors on the inventory cost. By varying the maximum support from 1 to 4, we solved the problem with the LOP formulation. With the obtained order policy from the LOP formulation, we also conducted simulation experiments 10 000 times for each instance. In addition, EV | PI was calculated by generating the uncertain factors with 10 000 times to see how the expected cost of stochastic programming differs from the approximation process as support increases. We present the computational results in Table VI. As can be observed from Table VI, the expected cost of the LOP formulation increases when the maximum support of the uncertain factor increases. The same trend was observed in the mean of the simulation studies. Since the AARC seeks the optimal solution with the worst-case expectation, the expected cost of LOP and simulation results show an increasing tendency when the supports of uncertain factors increase. The range of the uncertain demand, which the uncertain factors could realize, widens and makes the larger expected cost respond to the worst-case

TABLE VI
RESULTS OF EXPERIMENT 2: EFFECT OF THE SUPPORT OF UNCERTAIN FACTORS TO THE EXPECTED COSTS AND SIMULATION RESULTS

		Expected cost			Simulated inventory runs	
$\|\alpha\| \leq$	$\|z\| \leq$	EV|PI	LOP	Gap (%)	Mean	Gap (%)
0	1	51 074.8	51 253.4	0.35	51 438.8	0.71
	2	51 075.6	51 431.7	0.70	51 789.9	1.40
	3	51 073.8	51 610.0	1.05	52 149.6	2.11
	4	51 074.2	51 788.3	1.40	52 515.5	2.82
0.1	1	51 074.8	51 264.4	0.37	51 449.7	0.73
	2	51 076.0	51 453.6	0.74	51 812.8	1.44
	3	51 074.1	51 642.9	1.11	52 182.5	2.17
	4	51 074.0	51 832.2	1.45	52 558.4	2.91
0.2	1	51 074.2	51 285.7	0.41	51 488.3	0.81
	2	51 075.4	51 496.3	0.82	51 895.0	1.60
	3	51 072.8	51 706.8	1.24	52 298.5	2.40
	4	51 074.9	51 919.4	1.65	52 596.2	2.98
0.3	1	51 074.9	51 286.4	0.41	51 457.8	0.75
	2	51 075.5	51 497.6	0.83	51 832.6	1.48
	3	51 072.5	51 708.9	1.25	52 213.4	2.23
	4	51 072.7	51 923.1	1.67	52 430.8	2.66
0.4	1	51 074.4	51 403.1	0.64	52 562.6	2.91
	2	51 075.7	51 553.8	0.94	52 039.5	1.89
	3	51 073.9	51 794.1	1.41	52 458.2	2.71
	4	51 074.3	52 037.7	1.89	52 834.3	3.45
0.5	1	51 075.7	51 310.6	0.46	51 551.8	0.93
	2	51 077.6	51 546.2	0.92	52 018.7	1.84
	3	51 074.6	51 784.6	1.39	52 351.7	2.50
	4	51 076.3	52 051.4	1.91	53 015.0	3.80

TABLE VII
RESULTS OF EXPERIMENT 3: VERIFICATION OF EFFECT WITH OPERATING BOPIS AND OSHDS SIMULTANEOUSLY

$T = 20$	Expected cost			Simulated inventory runs		
$\|\alpha\| \leq$	BOPIS	OSHDS	BOPIS and OSHDS	BOPIS	OSHDS	BOPIS and OSHDS
0	50 751.5	58 156.9	48 607.8	50 749.2	58 217.3	49 024.4
0.1	50 795.6	58 226.7	48 641.1	50 792.6	58 274.2	49 041.8
0.2	50 901.9	58 354.4	48 708.5	50 898.4	58 398.9	49 168.9
0.3	50 878.9	58 342.2	48 724.3	50 862.1	58 391.7	49 108.2
0.4	51 011.3	58 421.9	48 836.0	50 919.6	58 484.7	49 281.7
0.5	51 096.7	58 601.2	48 868.1	50 997.5	58 549.2	49 250.9

distribution. In the case of the EV | PI, substantial fluctuations were not observed, even though the support increased. Consequently, gaps between the EV | PI and LOP, and EV | PI and the mean of the simulation studies become more prominent when the supports of the uncertain factors increase.

D. Experiment 3: Simultaneous Operations of BOPIS and OSHDS

We analyzed the advantages of simultaneous operations of BOPIS and OSHDS by conducting comparative experiments. We assumed the following three types of systems:

i) operating only BOPIS,
ii) operating only OSHDS, and
iii) BOPIS and OSHDS simultaneously.

To highlight the feature of systems (i) and (ii), we illustrate the framework in Figs. 5 and 6 in Appendix. In the case of (ii) OSHDS, the demand $\tilde{\xi}$ is not included. Instead, we assumed that this service option could carry the B&M demand. The comparative experiments were conducted based on the same input parameters, and results are tabulated in Table VII. We observe that the expected cost and mean of the simulated inventory runs of the simultaneous operations of the BOPIS and OSHDS show the lowest value among the three types of operations. In the instance we considered, the effect of BOPIS was larger than that of OSHDS. The operation cost was much higher when only OSHDS was considered, as compared to the operation cost of BOPIS alone. By excluding BOPIS, the retailer had to handle a large amount of demand in the B&M store, which naturally resulted in unsatisfied demand. BOPIS can play a role in increasing demand by attracting the attention of customers, but it also provides retailers with flexibility in terms of efficient inventory management.

TABLE VIII
RESULTS OF EXPERIMENT 4: ANALYSIS OF TENDENCY OF OSHDS WHEN BOPIS DEMAND IS DIVIDED INTO B&M STORE DEMAND AND ONLINE STORE DEMAND

$T = 20$	Expected cost				
β	$\gamma = 1$	$\gamma = 0.9$	$\gamma = 0.8$	$\gamma = 0.7$	$\gamma = 0.6$
1	64 379.2	62 132.4	59 885.5	57 638.7	55 391.8
0.75	63 087.2	60 969.5	58 851.9	56 734.2	54 616.6
0.5	61 795.1	59 806.7	57 818.2	55 829.8	53 841.4
0.25	60 503.1	58 643.8	56 784.6	54 925.4	53 066.1
0	59 211.0	57 481.0	55 751.0	54 020.9	52 290.9

We conducted additional experiments on how OSHDS behaves when BOPIS demand is divided into B&M store demand and online store demand. The previous experimental data were structured in a format in which B&M store demand and online store demand share the demand of BOPIS according to the ratio of β. In addition, an experiment was performed on the demand reduced by γ rather than all demand for BOPIS being transferred. That is, the demands of B&M and online stores are assumed to be as follows:

$$d_t \leftarrow d_t + \beta \cdot \gamma \cdot \xi_t$$
$$\phi_t \leftarrow \phi_t + (1 - \beta) \cdot \gamma \cdot \xi_t.$$

With the newly generated demands, we conducted additional experiments and results are tabulated in Table VIII. As can be observed from Table VIII, a decreasing tendency was shown when the value of β decreased. This tendency could be interpreted to mean that the unsatisfied demand of the online demand could be alleviated by the delayed delivery option. Although the unsatisfied demand of the B&M store demand could be handled by the OSHDS, the delayed option of the online store can be said to be more efficient in terms of inventory cost operation.

VI. CONCLUSION

A paradigm shift from the multichannel to the omnichannel system aids customers with their individual preferences and shopping convenience. From the perspective of the retailer, however, this shift makes controlling the inventory system more complicated due to the increase of the number of customer choices. In this article, we considered the BOPIS and OSHDS, which are representative examples of omnichannel characteristics. The opposing characteristics of the BOPIS and OSHDS options were incorporated in the mathematical formulations. Unlike the multichannel system, which manages inventory independently for each channel, the omnichannel system operates in an integrated manner. In other words, each channel's customer demand, which was assumed to be independent, should be regarded as integrated manner. The retailer should appropriately respond to customers through new ways of inventory control. Hence, we incorporated the correlation among uncertain demands by adopting the factor-based demand model. To handle the factor-based demand model, which makes the model intractable in a multistage decision process, we restricted decision variables to a linear function of the predefined uncertain factors. By utilizing the distributionally robust bound of the objective function, we derived the affinely adjustable robust counterpart, which features the deterministic second-order cone program.

A. Managerial Insights

This research provides managerial insights, which could be informative to inventory managers of omnichannel retailers. By conducting the computational experiments, we obtained the managerial insights as follows.

1) Although the demand model features a nonstationary process, it was observed that the gap between EV | PI and LOP is not large. Similar patterns were observed for simulated inventory runs. In other words, the approach considered in this article can be seen to provide a stable solution even when the variability of demand increases.
2) The inventory manager should be aware that if the support sets of uncertain factors were widely estimated, the confidence interval of the data could be increased, but a conservative inventory policy can also be obtained. In other words, the inventory manager should consider the tradeoff between the support set and conservatism.
3) Operating BOPIS and OSHDS complements each other with respect to the stable inventory policy. Both BOPIS and OSHDS not only function as promotions that induce purchases to customers, but operating them simultaneously could minimize the worst-case expected inventory cost.
4) In this omnichannel system, BOPIS plays a more important role than OSHDS does from an inventory cost perspective because it responds to its own demand directly. In the case of OSHDS, although it does not directly respond to customer demand, it plays a key role in maintaining the company's customers because it could prevent a demand loss by responding to stockout situations from uncertain demands. In other words, it can be said that the importance of OSHDS changes depending on how retailers estimate costs for demand loss.

B. Limitation of the Research and Future Study

This research has a limitation in that we only considered the retailer who operates one B&M store and one online store. Nevertheless, we believe that this research could be the basis of the omnichannel retailing inventory model, which considers the correlated uncertain demand. In other words, this article can serve as a cornerstone in extending research into the omnichannel retailing. If the single supplier, single manufacturer, and single retailer are generalized as the multisupplier, multimanufacturer, or multiretailer, the problem will be modeled as a more realistic problem. Also, we restricted some assumptions which are incorporated in the mathematical model. Although we restricted the fulfillment priority, as shown in Assumption 1, which stands for realistic assumptions in practice, some retailers could set fulfillment priority as a different order. Changing the order of fulfillment priority will enable the analysis of what each aspect will show in terms of total inventory cost. Furthermore, the following other types of omnichannel service can be considered as an example of future work. Some retailers recently launched a service called *buy online*, *return in store*. Since returns comprise a large proportion of sales, the ease of handling returns from customers is of utmost importance [66]–[68]. Returns, in fact, are much easier to handle within an omnichannel distribution system. Returns received in a warehouse will be inspected and quickly placed into inventory, regardless of where the stock originated. The same is true for returns received in a B&M store. Some researchers might consider the omnichannel system in future work that could include order-in-store-delivery-to-home, which is a more generalized concept of the out-of-stock home-delivery service. In this system, customer-switching behavior or customer-switching cost could be incorporated into the mathematical model. If these new problems are examined in an inventory model, we believe that such research would provide meaningful insights for the relevant retailers.

TABLE IX
PARAMETERS USED FOR THE COMPUTATIONAL EXPERIMENTS

d_t	$50+50\cdot U(0,1)$
p_t	$40+50\cdot U(0,1)$
ξ_t	$45+50\cdot U(0,1)$
c_t	$15+U(0,1)$
$h_{u,t}$	$1+U(0,1)$
$h_{v,t}$	$0.9+U(0,1)$
$h_{w,t}$	$0.8+U(0,1)$
$b_{u,t}$	$20+U(0,1)$
$b_{v,t}$	$20+U(0,1)$

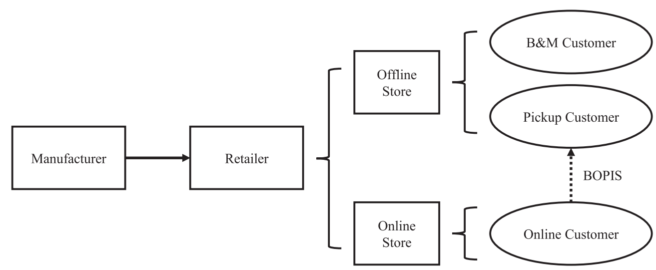

Fig. 5. Framework of BOPIS.

APPENDIX

We conducted comparative experiments on two types of systems in Section V-D: 1) a system that considered only BOPIS, and 2) a system that considered only OSHDS. We illustrated each framework in Figs. 5 and 6, respectively. For the comparative experiments, we developed a mathematical model for each

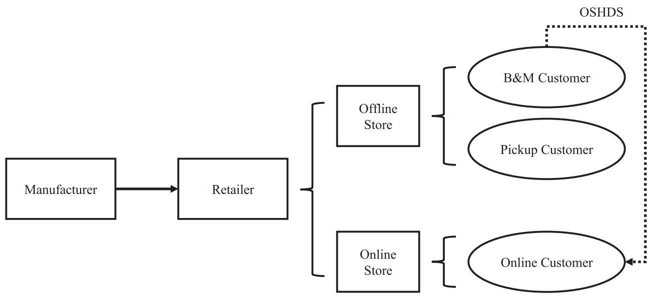

Fig. 6. Framework of OSHDS.

system. Since the development process of each deterministic formulation to AARC is the same as described in Section IV, we omitted the process. Each formulation is represented as follows:

$$\begin{aligned}
\min\quad & \mathrm{TOC} + \mathrm{TDC} + \mathrm{THC} + \mathrm{TBC} & \\
\text{s.t.}\quad & u_{t+1} = u_t + x_{t-L_m} - d_t & t \in \mathfrak{T} \\
& v_{t+1} = v_t + y_{t-L_m} - \xi_t & t \in \mathfrak{T} \\
& w_{t+1} = w_t + o_{t-L_m} + m_t - \phi_t & t \in \mathfrak{T} \\
& m_t = \sum_{k=1}^{\tau} m_{t,t-1+k} & t \in \mathfrak{T} \\
& (w_t + o_{t-L_m} - q_t - \phi_t)^- \geq m_t & t \in \mathfrak{T} \\
& x_t + y_t + o_t + \sum_{k=1}^{\tau} m_{t+1-k,t} \leq C_t & t \in \mathfrak{T} \\
& u_t + v_t \leq U_t & t \in \mathfrak{T}^+ \\
& w_t + o_{t-L_m} - \phi_t \leq K_t & t \in \mathfrak{T} \\
& x_t, y_t, o_t, \text{ and } m_t \geq 0 & t \in \mathfrak{T} \quad (39)
\end{aligned}$$

$$\begin{aligned}
\min\quad & \mathrm{TOC}^* + \mathrm{TIC} + \mathrm{TDC} + \mathrm{THC}^* + \mathrm{TBC}^* & \\
\text{s.t.}\quad & u_{t+1} = u_t + x_{t-L_m} - d_t + q_{t-L_r} & t \in \mathfrak{T} \\
& w_{t+1} = w_t + o_{t-L_m} + m_t - q_t - \phi_t & t \in \mathfrak{T} \\
& m_t = \sum_{k=1}^{\tau} m_{t,t-1+k} & t \in \mathfrak{T} \\
& (u_t + x_{t-L_m} - d_t)^- \geq q_{t-L_r} & t \in \mathfrak{T} \\
& (w_t + o_{t-L_m} - q_t - \phi_t)^- \geq m_t & t \in \mathfrak{T} \\
& x_t + o_t + \sum_{k=1}^{\tau} m_{t+1-k,t} \leq C_t & t \in \mathfrak{T} \\
& u_t \leq U_t & t \in \mathfrak{T}^+ \\
& w_t + o_{t-L_m} - q_t - \phi_t \leq K_t & t \in \mathfrak{T} \\
& x_t, o_t, m_t, \text{ and } q_t \geq 0 & t \in \mathfrak{T} \quad (40)
\end{aligned}$$

where

$$\mathrm{TOC}^* = \sum_{t \in \mathfrak{T}} \left[c_t \left(x_t + o_t \right) \right]$$

$$\mathrm{THC}^* = \sum_{t \in \mathfrak{T}} \left[h_{u,t}(u_{t+1})^+ + h_{w,t}(w_{t+1})^+ \right]$$

$$\mathrm{TBC}^* = \sum_{t \in \mathfrak{T}} \left[b_{u,t}(u_{t+1})^- \right].$$

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