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Carbon emission controlled investment and warranty policy based production inventory model via meta-heuristic algorithms

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ABSTRACT

Over the last few years, manufacturing firms are required to take more action in order to control the carbon emissions from their activities. Again, the warranty of a product is an important factor for both buyers and manufacturers. However, due to uncertain market situations and fluctuations of customers' demand, to conduct a detailed analysis of a manufacturing system is a complicated task. Primarily addressing these concepts together, in this work, an interval valued production inventory model is formulated under carbon taxation regulation in which demand is dependent on warranty period of the products. The main objective of this work is to determine the effect of warranty period of the products on the optimal policy of the production firm. Parallelly, this work also navigates the effect of carbon taxation regulation on the revenue of the manufacturer. There arises an interval optimization problem that is solved by centre-radius optimization technique. Further, to illustrate the validity of the model, a numerical example is considered and solved by different variants of quantum behaved particle swarm optimization (QPSO) techniques, grey-wolf optimizer algorithm (GWOA), teaching learning based optimizer algorithm (TLBOA), sparrow search algorithm (SSA). From the findings of numerical solution, it is observed that all the algorithms are equally efficient. Sensitivity analysis indicates that the centre of the average profit of the system increases most significantly as the initial demand rate of the products increases. Further, the warranty period of the products affects the optimal policy of the system in a suffice way and carbon taxation affects the revenue of the system less significantly. Finally, as a practical illustration of the model, another numerical example is taken into account considering the manufacturing and business strategies of LED monitor in a local manufacturing firm of Kolkata (India).

1. Introduction

Within manufacturing firms, it is essential to produce perfect quality of items during the production process. However, in most of the cases, it is observed that a small percentage of the whole production appears to be defective. There are several factors behind the production of some defective products, viz. efficiency of machines, capability of workers and operators, quality of supplied raw materials, certain natural calamity, uncertain power cuts in electric supply and some other factors. Subsequently, the defective products are either rejected or reworked, to convert them into perfect quality of products or to sell them at lower prices in their defective states. In the manufacturing industry, generally, there exist three departments: Marketing department (MD), Production department (PD) and Research & Development (R&D) department. Generally, the marketing department surveys the market to take appropriate action to meet the demands of consumers. Meanwhile, the R&D department endeavours to minimize the imperfect rate of production and production costs, update the production processes and introduce the updated and newly fabricated products. Production department then start to produce the items after taking the decisions of Marketing and R&D departments, with the goal to minimize the overall cost of production in order to maximize the average profit of the system. In this connection, Rout et al. (2019) studied the optimal policy of a defective production system of deteriorating items. Furthermore, the

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defective production problem has been extended to various directions. Panja and Mondal (2019) analysed the defective production process in the four-layer supply chain model. Malik and Sarkar (2020) considered disruption management in a constrained multi-product defective production system. Maiti (2021) extended the defective production problem by considering production rate dependent demand and Das et al. (2022) introduced the effect of SAR of products on demand in their proposed defective production model. Kishore et al. (2022), Khara et al. (2021), Singh and Chaudhary (2021), Alfares and Ghaithan (2022), Noblesse et al. (2022), Bhawaria and Rathore (2022), Keller et al. (2022) and others introduced various types of business dimensions in their proposed defective production inventory/supply chain models.

The customers' demand rate of any product plays one of the most vital roles in inventory/production inventory as it reflects the depletion rate of products from the warehouse/production house to meet the customers' desires. Several factors, viz. selling price, warranty period, quality, stock-level, frequency of advertisement, discounts, green level and brand value have little or great impact on customers' demand. According to the principle of demand in economics, the selling price and demand of an item follow an antagonistic relationship when all other factors remain the same. i.e., an item with a lower selling price is much more popular than the same item with a higher selling price. Therefore, it is an essential need for every manufacturer to control the production cost so that it helps to control the selling price of the products. In this area, several works have been done considering price dependent demand rate. Sarkar et al. (2014) modelled an economic manufacturing quantity (EMQ) system considering price and time sensitive demand rate. Maiti and Giri (2015) extended this work to a closed loop supply chain by considering demand rate as a linear function of selling price and quality of the products. Thereafter, Alfares and Ghaithan (2016) proposed an inventory model with demand rate as a linear decay function of price. Khanna et al. (2017) incorporated the selling price dependent non-linear demand in the inventory model of deteriorative items. Hu et al. (2018) extended this work into a supply chain coordination under demand as a decreasing function of a selling price. After that, Pervin et al. (2019) developed a multi-item inventory model with price and stock sensitive demand rate. Furthermore, Giri and Masanta (2020) developed a closed loop supply chain considering price and stock dependent demand rate. Halim et al. (2021) studied an overtime production inventory model for deteriorating items with non-linear price and stock dependent demand rate. Recently, Das et al. (2022) analysed a production inventory model with price sensitive demand rate using the theory of interval valued optimal control.

Nowadays, warranty period of the products is one of the attractive business strategies for various products, viz. cell phones, TV, refrigerator, any kind of electronics goods and automobiles. Customers are generally interested about the warranty policy at the time of purchasing the product as they believe that the warranty based product is more reliable and it has high longevity. This reason prevails a protagonist relationship between products' warranty period and customers' demand. Parallelly, in order to prefer the warranty period, manufacturers are bound to use good quality/quality certified raw materials. As a result, unit production cost will increase and it directly depends on the warranty period. Few works have been accomplished by considering the warranty period dependent demand and production cost in the area of production inventory system. Manna et al. (2020) studied a production inventory model considering warranty dependent demand and production cost. Khanna et al. (2020) studied the warranty policy and maintenance strategy by setting an integrated vendor–buyer supply chain. Guchhait et al. (2020) extended a defective production inventory system considering the warranty policy and investment for process quality improvement. Hou et al. (2021) incorporated the warranty period in their purchase model. Manna et al. (2021) navigated optimal policy of a two-plant production model with the warranty period dependent demand and Samanta & Giri (2021) applied the concept of pro-rata warranty policy in their supply chain model. Recently, Yazdian et al. (2016), Chen et al. (2017), Keshavarz and Arshadi (2022), Manna and Bhunia (2022), Utama et al. (2022) analysed the effect of the warranty period on the optimal pricing strategy.

During the production process, manufacturing firms emit greenhouse gases (viz. carbon dioxide, sulphur dioxide, carbon monoxide, and CFCs) which are responsible for global warming and uncertain climate change. Because of global warming, the average temperature of the Earth is steadily rising, resulting in rapid melting of Arctic ice. Paradoxically, climate change causes due to natural calamities like floods, droughts, storms, and wildfires. As an essential requirement, the governments of most countries are actively working to reduce carbon emissions and they have formally applied carbon emissions rules to manufacturing systems. In this situation, one of the main goals of production systems is to minimize carbon emissions and environmental pollution during production. Manufacturers have to invest in reducing emissions. Lin and Sarker (2017) considered carbon emissions reduction policy in a pull system inventory model of defective quality items. Zadjafar and Gholamian (2018) and Shen et al. (2019) investigated the optimal decisions of a manufacturing model considering carbon tax. Lu et al. (2020) and Lu et al. (2020) formulated a multi-stage sustainable production model considering the emissions reduction effort. Shi et al. (2020) developed various types of production inventory models that take carbon tax into account. Sepehri et al. (2021) studied a defective production inventory model under preservation technology, considering carbon emissions reduction effects on the optimal policy. Furthermore, Jauhari et al. (2021) investigated optimal policy of a closed-loop supply chain model that took into account hybrid production processes, take-back incentives and carbon emissions. Manna et al. (2021) and Das et al. (2022) considered different types of production inventory models considering carbon emissions investments/taxations. Beside these, the works of Saga et al. (2019), Wee and Daryanto (2020), Rout et al. (2020, 2021), Karthick and Uthayakumar (2021), Ruidas et al. (2021), and Das et al. (2020) are worth mentioning. A comparative review related to the proposed work is shown in Table 1.

Optimization problems related to inventory models entail either the maximization of average profits or the minimization of the average cost of the model, subject to certain conditions. Therefore, in studying optimization techniques, it is essential to analyse the optimal policy of an inventory model. Their optimization methods are fundamentally divided into two categories:

- Traditional optimization technique
- Non-traditional optimization technique

In the first method, the derivative information of the objective functions is required. However, most of the real-world optimization problems have objective functions that are very complicated and non-linear in nature. In many of these cases, the objective functions are even non-differentiable. In this case, traditional optimization methods fail to determine optimal solution. For this reason, non-traditional optimization techniques are needed. Non-traditional techniques require no derivative information of the objective functions. Metaheuristic algorithms are one of the non-traditional optimization techniques. The metaheuristic algorithms are categorised into four major categories (see Table 2):

- Evolutionary based algorithms
- Swarm intelligence based algorithms
- Human-based algorithms
- Nature-based algorithms.

To tackle the uncertainty in a real-life problem is a difficult task. To overcome this difficulty, several approaches (specifically, stochastic, fuzzy, fuzzy-stochastic and interval) are used. Among these, the obtained results of any imprecise mathematical problems under interval uncertainty are much more understandable than the results obtained

Table 1
Comparative review related to the proposed model.

Models proposed by	Type of model	Customers' demand dependent on		Warranty policy	Carbon investment policy	Type of uncertainty	Solution technique
		Selling price	Warranty period				
Cao et al. (2017)	Purchase	√	×	×	√	Crisp	Analytical method
Zhang and Liu (2018)	Production	×	×	×	√	Crisp	Game theory approach
Zhang et al. (2019)	Production	×	×	√	×	Crisp	Analytical method
Shaikh et al. (2019)	Production	√	×	×	×	Interval	PSO algorithm
Taleizadeh et al. (2018)	Purchase	×	×	√	×	Crisp	Hybrid NSGA-II
Das et al. (2020)	Production	√	√	√	×	Crisp	MATHEMATICA
Mishra et al. (2020)	Production	×	×	×	√	Crisp	Analytically
Lu et al. (2020)	Production	×	×	×	√	Crisp	Stackelberg game
Manna et al. (2020)	Production	×	√	√	×	Crisp	MATHEMATICA
Jauhari et al. (2020)	Supply chain	√	×	×	√	Crisp	Stackelberg game
Rout et al. (2021)	Production	×	×	×	√	Crisp	VNS algorithm
Ruidas et al. (2021)	Production	√	×	×	√	Interval	QPSO algorithm
Samanta and Giri (2021)	Supply chain	√	√	√	×	Crisp	Analytical method
Hou et al. (2021)	Production	×	×	√	×	Crisp	Matrix analytical method
Jauhari and Wangsa (2022)	Production	×	×	×	√	Probabilistic	Analytical method
Paul et al. (2022)	Purchase	√	×	×	√	Crisp	MATHEMATICA
Maji et al. (2022)	Production	√	√	√	×	Probabilistic	NSGA-II
Das et al. (2022)	Production	√	×	×	×	Interval	*c-r* optimization technique and variants of QPSO
Present authors	**Production**	√	√	√	√	**Interval**	***c-r* optimization technique, variants of QPSO algorithms and GWOA, TLBOA, SSA**

under fuzzy or probabilistic techniques. In inventory management, the price of products and different costs lie within a range. We have used this range as an interval. On the other hand, if a parameter is considered to be a fuzzy set or a fuzzy number, then we have to select the appropriate membership function or type of fuzzy number. Also, in stochastic cases, we have to choose the appropriate probability distribution function. Selecting an appropriate membership function or type of fuzzy number or probability distribution is a difficult task. Furthermore, most of the studies mentioned above did not cover the impact of the warranty period and carbon reductions investment together to the optimal policy of a production system. Those who incorporated the effect of warranty, did not consider the effect of carbon taxation and vice-verse. Furthermore, very few works have been done in the area of production inventory under interval uncertainty. Primarily, motivating from these facts, for the first time, we have proposed a production inventory model under interval uncertainty considering the impacts of both warranty as well as carbon reduction investment on the optimal pricing strategy.

1.1. Research questions and contribution

After a deep survey of existing literature in the field of production inventory, some research gaps (cf. Table 1) have been found. These gaps can be summarized as the following research questions:

(i) How the revenue of a production system is affected due to maintain the warranty period of products?
(ii) Which strategies should be adopted by the authority of a manufacturing firm so that it helps to generate more revenue under the impact of emission reduction constrain.
(iii) How to deal with the uncertain situation of market economy?
(iv) How to optimize the interval valued highly non-linear optimization problem (i.e., average profit of the manufacturing firm)?

To fill up these gaps, a production inventory model has been proposed in which interval valued customers' demand is dependent linearly on the selling price and warranty period of the products. Also, the carbon reduction technology has been considered to reduce the environmental pollution due to emission of carbon dioxide, carbon monoxide, and several other greenhouse gases during the production process.

Thus, the main contributions of this manuscript are summarized below:

(i) Customer demand is considered as an interval-valued function depending on the selling price and the warranty period of the product. Also, manufacturer needs to pay due to warranty which depends linearly on warranty period of the products.

Table 2
Some existing meta-heuristic algorithms.

Evolutionary-based metaheuristic algorithms	Human-based metaheuristic algorithms	Nature-based metaheuristic algorithms	Swarm intelligence-based metaheuristic algorithms
Genetic Algorithm (GA) (Holland (1975)), Differential Evolution (DE) (Storn and Price (1997)), Evolutionary Programming (EP) (Cao and Wu (1997)), Tournament GA (TGA) (Yang and Soh (1997)), Real-Coded GA (RCGA) (Blanco et al. (2001))	Dynamic Hill Climbing Algorithm (Yuret and De La Maza (1993))The Flower Pollination Algorithm (FPA) (Yang (2012)), Teaching Learning Based Optimization Algorithm (TLBOA) (Rao (2016)) Human Mental Search Algorithm (HMSA) (Mousavirad and Ebrahimpour-Komleh (2017))	Grey Wolf Optimizer Algorithm (GWOA) (Mirjalili et al. (2014)), Ant Lion Optimizer Algorithm (ALOA) (Mirjalili (2015)), Whale Optimizer Algorithm (WOA) (Mirjalili and Lewis (2016)), Sparrow Search Algorithm (SSA) (Xue and Shen (2020))	Particle Swarm Optimization (PSO) (Kennedy and Eberhart (1995)), Ant-Colony Optimization (ACO) (Dorigo et al. (2006)), Bee-Colony Optimization (BCO) (Teodorovic et al. (2006)), Quantum behaved Particle Swarm Optimization (QPSO) (Sun et al. (2012))

(ii) Carbon emissions reduction investment is considered as linear interval valued function of time.
(iii) The model is developed in an interval environment using parametric approaches of intervals and interval differential equation.
(iv) The optimality conditions for the model in interval environment are exaggerated by the centre-radius optimization (or *c-r* optimization) technique. Further, the considered numerical examples are solved using various meta-heuristic algorithms (viz. AQPSO, GQPSO, WQPSO, GWOA, SSA, TLBOA etc.).

The rest of the paper is organised as follows:

Problem description is presented in Section 2 with its fundamental notations and necessary assumptions. The model is formulated mathematically in Section 3. The solution methodology part is addressed in Section 4 with the description of meta-heuristic algorithms and *c-r* optimization technique. Section 5 is comprised with a numerical example and a practical example. The sensitivity analyses are performed in Section 6 and managerial insights are discussed in Section 7. Finally, some concluding remarks are noted in Section 8.

2. Problem description

Due to fluctuations in the market economy and growth of the gross domestic product of countries, uncertainty is an important factor in every sector of business. In any inventory control, the values of different parameters, such as demand and production rates, different cost factors and the selling price of the product are imprecise and uncertain. To represent the uncertainty of different parameters, interval approach is used. Meanwhile, one of the attractive business strategies, i.e., warranty policy, is adopted by the manufacturer. Through the warranty policy, customers are assured about the quality and reliability of the products, and hence they are enthusiastic to purchase them. As a result, demand is related to the warranty period of the products. The selling price of products may impose a negative impact on customers' demand i.e., if price increases, then demand of the corresponding product decreases. By relating these real cases to demand, customers' demand is considered as a linearly increasing function of the warranty period of the products and a linear decay function of time. Again, to enhance the reliability of the product and also to enhance customers' demand, manufacturer provides warranty period along with a free service during the warranty period of the products. On the environmental front, controlling of carbon emissions is difficult during the production and transportation of goods. This is true in every country. Some developed countries have imposed rules and regulations for manufacturing firms to control carbon emissions, and because of this, manufacturing firms must invest to reduce the carbon emissions during the production period and its rate is taken as a linear increasing function of time. For this reason, we have considered both manufacturers' carbon emissions control investments and product warranty policies in this study. The graphical representation of the problem is shown in Fig. 1.

The notation and assumptions regarding the proposed production model are summarized in Sections 2.1 and 2.2 respectively.

2.1. Notation

Notation:	Description
$[I_L(t), I_U(t)]$	Inventory level at time t(unit)
$[P_{rL}, P_{rU}]$	Production rate (unit/year)
$[\theta_{rL}, \theta_{rU}]$	Defective rate, $0 < \theta_{rL} < \theta_{rU} << 1$
T	Business period (year)
$[d_L(s_p, w_p), d_U(s_p, w_p)]$	Demand rate (unit/year)
$[c_L, c_U]$	Unit production cost ($/unit)
$[h_{cL}, h_{cU}]$	Holding cost/ unit/ unit time ($/unit/year)
$[A_L, A_U]$	Setup cost per cycle ($/order)
$[I_{Lco_2}(t), I_{Uco_2}(t)]$	Carbon emissions investment rate at time t ($/year)
t_p	Production period (year) (decision variable)
w_p	Warranty and free service period of the product (year) (decision variable)
s_p	Selling price per unit item ($/unit) (decision variable)
$[\pi_L(t_p, w_p, s_p),\ \pi_U(t_p, w_p, s_p)]$	Average profit ($/year)
$\langle \pi_c(t_p, w_p, s_p), \pi_r(t_p, w_p, s_p) \rangle$	Centre, radius of interval-valued average profit ($/year)

2.2. Assumptions

(i) The proposed work deals with a defective production model in which a part of the produced items is defective.

(ii) The production rate of the manufacturing firm is lying in a constant interval $[P_{rL}, P_{rU}]$. This function is commonly applied in the existing literature of Rahaman et al. (2020) and Manna and Bhunia (2022).

(iii) The defective rate of production is considered as $[\theta_{rL}P_{rL}, \theta_{rU}P_{rU}]$ in an interval environment.

(iv) Customers are generally interested in the warranty policy when they purchase products as they believe that the warranty based product is more reliable and it has high longevity. Also customers like long warranty period that means if the warranty period is long then the demand is increased. On the other hand, if the selling price of a product increases then demand of the corresponding product decreases. Combining these facts, customers' demand rate is taken as a linear interval-valued function of the selling price and warranty period of the product. According to Giri et al. (2018), the mathematical form of the demand rate is given by

$$[d_L(s_p, w_p), d_U(s_p, w_p)] = [\alpha_L, \alpha_U] + [\beta_L, \beta_U]w_p - [\gamma_L, \gamma_U]s_p$$

where $[\alpha_L, \alpha_U]$ is the location parameter of the demand. On the other hand, $([\beta_L, \beta_U],\ [\gamma_L, \gamma_U])$ are the shape parameters of demand where $\alpha_L > 0, \beta_L > 0, \gamma_L > 0$.

(v) The manufacturer offers a warranty along with a free service period to the customers. Due to the free service and warranty of such products, the servicing cost per unit product is given by

$$[s_L(w_p), s_U(w_p)] = [\mu_L, \mu_U]w_p$$

(vi) For controlling carbon emissions during the production period, the manufacturing authority invests an amount per unit of production. Generally, as the production period of a manufacturing system in-

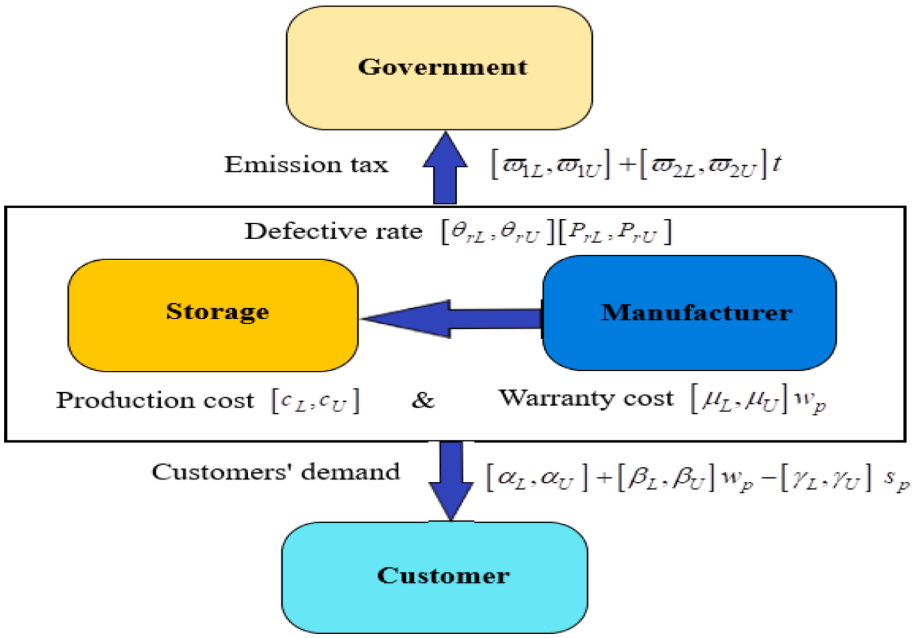

Fig. 1. Graphical representation of the problem mentioned in Section 2.

creases, the emissions rate of the firm increases due to various factors, such as inefficiency and low motivation of workers in the manufacturing firm, the reduction of the suction ability of the filters connected to the chimney, and other factors. For these reasons, to amend the emissions level of the manufacturing firm, the authority of the firm must invest with a rate dependent on linear time (i.e., per unit investment per time increases deliberately) within the production period i.e. within $[0,t_p]$. According to Manna et al. (2021) and Das et al. (2022), the emission reduction investment per year is considered as

$[I_{Lco_2}(t),I_{Uco_2}(t)] = [\varpi_{1L},\varpi_{1U}] + [\varpi_{2L},\varpi_{2U}]t,\ \ 0<t\leqslant t_p$where $[\varpi_{1L},\varpi_{1U}]$ represents the location parameters of the interval-valued carbon emissions investment function. On the other hand, $[\varpi_{2L},\varpi_{2U}]$ represents the shape parameters of carbon emissions investment function with $\varpi_{1L}, \varpi_{2L}>0$.

(vii) The planning horizon of this model is infinite and lead time is constant.

(viii) Shortages are not allowed.

3. Model formulation

Considering the imprecision of different inventory parameters, a production inventory model is formulated in the interval environment. According to the assumption, the manufacturing firm starts to produce items at time $t=0$ at an interval rate $[P_{rL},P_{rU}]$. During the production period, the interval-valued defective rate of the product is $[\theta_{rL},\theta_{rU}][P_{rL},P_{rU}]$. After fulfilling the customer demands, the stock of excess items increases at an interval-valued rate $(1-[\theta_{rL},\theta_{rU}])[p_{rL},p_{rU}]-[d_L(s_p,w_p),d_U(s_p,w_p)]$, up to the time $t=t_p$. After that, during the time interval, $[t_p,T]$, both bounds of the inventory level decline in order to meet the demand only and it reaches zero at time $t=T$. Therefore, during the business period $[0,T]$, the rate of change of the interval-valued inventory level is governed by the differential equations as follows:

$$\frac{d[I_L(t),I_U(t)]}{dt} = (1-[\theta_{rL},\theta_{rU}])[P_{rL},P_{rU}]-[d_L(s_p,w_p),d_U(s_p,w_p)],\qquad 0\leqslant t\leqslant t_p \tag{1}$$

$$\frac{d[I_L(t),I_U(t)]}{dt} = -[d_L(s_p,w_p),d_U(s_p,w_p)],\qquad t_p<t\leqslant T \tag{2}$$

with the boundary conditions

$$I_L(0)=0,I_U(0)=0,\ I_L(T)=0,I_U(T)=0. \tag{3}$$

At time $t\in[0,T]$, the status of inventory level is shown graphically in Fig. 2.

From Proposition A.1, the system of interval differential equations (1)–(2) is equivalent to their parametric forms as follows:

$$\frac{dI(t,\lambda)}{dt} = \{1-\theta_r(\lambda_3)\}P_r(\lambda_1)-d(s_p,w_p,\lambda_2),\ 0\leqslant t\leqslant t_p \text{ and } \lambda,\lambda_1,\lambda_2,\lambda_3\in[0,1] \tag{4}$$

$$\frac{dI(t,\lambda)}{dt} = -d(s_p,w_p,\lambda_2),\quad t_p<t\leqslant T \text{ and } \lambda,\lambda_2\in[0,1] \tag{5}$$

$$\text{where } I(t,\lambda)=I_L(t)+\lambda(I_U(t)-I_L(t)),\ P_r(\lambda_1)=P_{rL}+\lambda_1(P_{rU}-P_{rL}),\ \theta(\lambda_3)=\theta_L+\lambda_3(\theta_U-\theta_L) \text{ and } d(s_p,w_p,\lambda_2)=d_L(s_p,w_p,\lambda_2)+\lambda_2(d_U(s_p,w_p,\lambda_2)-d_L(s_p,w_p,\lambda_2)) \text{ with } I(0,\lambda)=I(T,\lambda)=0,\ \ \lambda\in[0,1]. \tag{6}$$

Solving (4)–(5) with the help of (6), the inventory levels at time t during production and non-production periods are as follows:

$$I(t,\lambda)=\{(1-\theta_r(\lambda_3))P_r(\lambda_1)-d(s_p,w_p,\lambda_2)\}t,\ 0\leqslant t\leqslant t_p \text{ and } \lambda,\lambda_1,\lambda_2,\lambda_3\in[0,1] \tag{7}$$

$$I(t,\lambda)=d(s_p,w_p,\lambda_2)\ (T-t),\ t_p<t\leqslant T \text{ and } \lambda,\lambda_2\in[0,1] \tag{8}$$

Using the parametric representation of intervals, using (7)–(8), the interval-valued inventory level can be written as

$$[I_L(t), I_U(t)] = \left[\left((1-\theta_{rU})p_{rL} - d_U(s_p, w_p)\right)t, \left((1-\theta_{rL})p_{rU} - d_L(s_p, w_p)\right)t\right],\ 0 < t \leqslant t_p \tag{9}$$

$$[HC_L(t_p, w_p, s_p), HC_U(t_p, w_p, s_p)] = [h_{cL}, h_{cU}]\left(\int_0^{t_p}[I_L(t), I_U(t)]\,dt + \int_{t_p}^{T}[I_L(t), I_U(t)]\,dt\right) = \left[\begin{array}{c} \frac{h_{cL}}{2}\left\{\left((1-\theta_{rU})P_{rL} - d_{rU}(s_p, w_p)\right)t_p^2 + d_{rL}(s_p, w_p)\,(T-t_p)^2\right\}, \\ \frac{h_{cU}}{2}\left\{\left((1-\theta_{rL})P_{rU} - d_{rL}(s_p, w_p)\right)t_p^2 + d_{rU}(s_p, w_p)(T-t_p)^2\right\} \end{array}\right] \tag{18}$$

$$[I_L(t), I_U(t)] = \left[d_L(s_p, w_p)\,(T-t), d_U(s_p, w_p)\,(T-t)\right],\ t_p < t \leqslant T \tag{10}$$

Combining (9) and (10), the lower and upper bounds of the inventory levels are as follows:

$$I_L(t) = \begin{cases} \left\{(1-\theta_{rU})p_{rL} - d_U(s_p, w_p)\right\}t, & 0 < t \leqslant t_p \\ d_L(s_p, w_p)\,(T-t), & t_p < t \leqslant T \end{cases} \tag{11}$$

$$I_U(t) = \begin{cases} \left\{(1-\theta_{rL})p_{rU} - d_L(s_p, w_p)\right\}t, & 0 < t \leqslant t_p \\ d_U(s_p, w_p)\,(T-t), & t_p < t \leqslant T \end{cases} \tag{12}$$

From the continuity of (11) at $t = t_p$, we get

$$(1-\theta_{rU})p_{rL}t_p = d_L(s_p, w_p)T \tag{13}$$

From the continuity of (12) at $t = t_p$, we get

$$(1-\theta_{rL})p_{rU}t_p = d_U(s_p, w_p)T \tag{14}$$

Combining (13) and (14), we have the following relationship

$$T = \frac{(1-\theta_{rL})P_{rU} + (1-\theta_{rU})P_{rL}}{d_U(s_p, w_p) + d_L(s_p, w_p)}t_p \tag{15}$$

The total interval-valued sales revenue is as follows:

$$[SR_L(t_p, s_p), SR_U(t_p, s_p)] = s_p\int_0^{t_p}(1-[\theta_{rL}, \theta_{rU}])[P_{rL}, P_{rU}]\,dt = \left[s_p(1-\theta_{rU})P_{rL}t_p,\ s_p(1-\theta_{rL})P_{rU}t_p\right] \tag{16}$$

The total interval-valued production cost is as follows:

$$[PC_L(t_p), PC_U(t_p)] = [c_L, c_U]\int_0^{t_p}[P_{rL}, P_{rU}]\,dt = \left[c_LP_{rL}t_p,\ c_UP_{rU}t_p\right] \tag{17}$$

The total interval-valued holding cost is given by

The total interval-valued servicing cost can be calculated as

$$[SC_L(t_p, w_p), SC_U(t_p, w_p)] = [s_L(w_p), s_U(w_p)]\int_0^{T}[d_L(s_p, w_p), d_U(s_p, w_p)]\,dt = [\mu_L, \mu_U]w_p[d_L(s_p, w_p), d_U(s_p, w_p)]T \tag{19}$$

The total interval − valued setup cost per cycle is given by $[A_L, A_U]$. (20)

The total interval-valued carbon emissions control investment is as follows:

$$[TI_{Lco_2}(t_p), TI_{Uco_2}(t_p)] = \int_0^{t_p}[\varpi_{1L} + \varpi_{2L}t, \varpi_{1U} + \varpi_{2U}t]\,dt = \left[\varpi_{1L}t_p + \frac{\varpi_{2L}}{2}t_p^2, \varpi_{1U}t_p + \frac{\varpi_{2U}}{2}t_p^2\right] \tag{21}$$

The interval-valued total profit per cycle (using (16)-(21)) is given by *Interval-valued total profit = <Interval-valued sales revenue>-<Interval-valued production cost>-<Interval-valued holding cost>-<Interval-valued total carbon emission investment>-<Interval-valued ordering cost>-<Interval-valued service cost>.*

$$\text{or, } [TP_L(t_p, w_p, s_p), TP_U(t_p, w_p, s_p)] = \left[\begin{array}{c} \left\{SR_L(t_p, s_p) - PC_U(t_p, w_p) - HC_U(t_p, w_p, s_p) - TI_{Uco_2}(t_p) - A_U - SC_U(t_p, w_p)\right\}, \\ \left\{SR_U(t_p, s_p) - PC_L(t_p, w_p) - HC_L(t_p, w_p, s_p) - TI_{Lco_2}(t_p) - A_L - SC_L(t_p, w_p)\right\} \end{array}\right]$$
$$= \left[\begin{array}{c} s_p(1-\theta_{rU})P_{rL}t_p - c_UP_{rU}t_p - \frac{h_{cU}}{2}\left\{\left((1-\theta_{rL})P_{rU} - d_{rL}(s_p, w_p)\right)t_p^2 + d_{rU}(s_p, w_p)(T-t_p)^2\right\} - \varpi_{1U}t_p - \frac{\varpi_{2U}}{2}t_p^2 - A_U \\ -\mu_Uw_pd_U(s_p, w_p)T, \\ s_p(1-\theta_{rL})P_{rU}t_p - c_LP_{rL}t_p - \frac{h_{cL}}{2}\left\{\left((1-\theta_{rU})P_{rL} - d_{rU}(s_p, w_p)\right)t_p^2 + d_{rL}(s_p, w_p)(T-t_p)^2\right\} - \varpi_{1L}t_p - \frac{\varpi_{2L}}{2}t_p^2 - A_L \\ -\mu_Lw_pd_L(s_p, w_p)T \end{array}\right] \tag{22}$$

Therefore, using (21), the interval-valued average profit is of the form

$$\left[\pi_L\left(t_p, w_p, s_p\right), \pi_U\left(t_p, w_p, s_p\right)\right] = \frac{1}{T}\left[TP_L\left(t_p, w_p, s_p\right), TP_U\left(t_p, w_p, s_p\right)\right] = \left[\frac{TP_L\left(t_p, w_p, s_p\right)}{T}, \frac{TP_U\left(t_p, w_p, s_p\right)}{T}\right]$$
$$= \left[\frac{\left(d_{rU}(s_p, w_p) + d_{rL}(s_p, w_p)\right)TP_L\left(t_p, w_p, s_p\right)}{((1-\theta_{rL})P_{rU} + (1-\theta_{rU})P_{rL})t_p}, \frac{\left(d_{rU}(s_p, w_p) + d_{rL}(s_p, w_p)\right)TP_U\left(t_p, w_p, s_p\right)}{((1-\theta_{rL})P_{rU} + (1-\theta_{rU})P_{rL})t_p}\right] \quad (23)$$
$$\text{using (15), putting the value of } T$$

Therefore, in this study, the objective is to determine the best-found values of t_p, w_p and s_p which maximize the interval-valued average profit (i.e., $\left[\pi_L\left(t_p, w_p, s_p\right), \pi_U\left(t_p, w_p, s_p\right)\right]$). Hence, the corresponding optimization problem is of the following form:

$$\begin{array}{ll} Maximize & \left[\pi_L\left(t_p, w_p, s_p\right), \pi_U\left(t_p, w_p, s_p\right)\right] \\ \text{subject to} & t_p > 0, w_p > 0,\ s_p > 0. \end{array} \quad (24)$$

4. Solution methodology

Clearly, the optimization problem (24) is a constrained interval-valued optimization problem. To solve this optimization problem, firstly we have to consider an interval ranking on the set of all compact intervals. Then, with respect to the proposed interval ranking, we need to propose the definition of an optimizer of an interval-valued optimization problem. Finally, regarding the definition of an optimizer of an interval-valued optimization problem, we need to solve (24).

To solve (24), we have considered the interval ranking proposed by Bhunia and Samanta (2014). This is a complete ordering that is based on centre-radius representations of intervals. Therefore, centre-radius representations of intervals, interval ranking and the definition of the maximizer of (24) are discussed below:

Definition 1: Let $[a_L, a_U] \in K_c$. Then the centre-radius representation of $[a_L, a_U]$ is defined by $\langle a_c, a_r \rangle$, where $a_c = \frac{a_L + a_U}{2}$ and $a_r = \frac{a_U - a_L}{2}$.

Definition 2: (Bhunia and Samanta, 2014). Let I_1 and I_2 be two intervals of real numbers such that $I_1 = [a_L, a_U] \cong \langle a_c, a_r \rangle$ and $I_2 = [b_L, b_U] \cong \langle b_c, b_r \rangle$, then the interval order relations between I_1 and I_2 are as follows:

(i) Ordering for the minimization problem: $I_1 \leqslant_{\min} I_2 \Leftrightarrow \begin{cases} a_c < b_c \\ a_r \leqslant b_r \text{ if } a_c = b_c \end{cases}$

(ii) Ordering for the maximization problem: $I_1 \geqslant_{\max} I_2 \Leftrightarrow \begin{cases} a_c > b_c \\ a_r \leqslant b_r \text{ if } a_c = b_c \end{cases}$

Definition 3: A point $\left(t_p^*, w_p^*, s_p^*\right) \in (0, \infty) \times (0, \infty) \times (0, \infty) \subset \mathbb{R}^3$ is said to be a global maximizer of the interval optimization problem (23) if it satisfies the following:

$$\left[\pi_L\left(t_p, w_p, s_p\right), \pi_U\left(t_p, w_p, s_p\right)\right] \geqslant_{\max} \left[\pi_L\left(t_p^*, w_p^*, s_p^*\right), \pi_U\left(t_p^*, w_p^*, s_p^*\right)\right], \text{ for all } \left(t_p, w_p, s_p\right) \in (0, \infty) \times (0, \infty) \times (0, \infty).$$

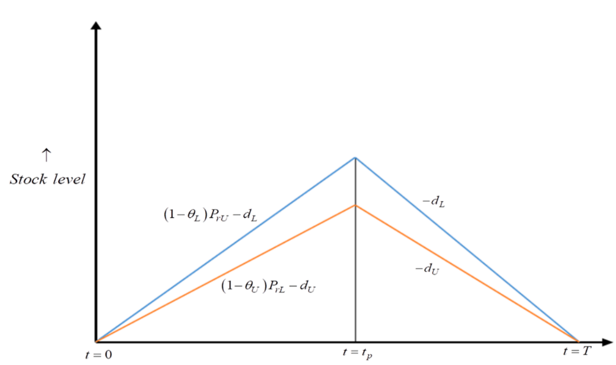

Fig. 2. Changes of (upper and lower) bounds of inventory level with respect to time.

The centre-radius optimization (*c-r* optimization) technique is applied to solve the maximization problem (24) with an interval-valued objective function. This optimization technique is based on the interval order relations proposed by Bhunia and Samanta (2014). In this technique, the interval maximization problem is solved either by maximizing the centre of the interval objective function (if not constant), subject to the constraints, or by minimizing the radius of the same. Therefore, this technique converts an interval optimization problem into a crisp optimization problem.

Proposition 1: ***(c-r optimization technique)*** (Rahman et al. (2020)). The interval average profit function.

$\left[\pi_L\left(t_p, w_p, s_p\right), \pi_U\left(t_p, w_p, s_p\right)\right] \equiv \left\langle \pi_c\left(t_p, w_p, s_p\right), \pi_r\left(t_p, w_p, s_p\right) \right\rangle$ has a maximizer at $t_p = t_p^*, w_p = w_p^*, s_p = s_p^*$.

if and only if

$$\begin{cases} \left(t_p^*, w_p^*, s_p^*\right) \text{ is a maximizer of } \pi_c\left(t_p, w_p, s_p\right) \text{ if } \pi_c\left(t_p, w_p, s_p\right) \neq \text{constant} \\ \left(t_p^*, w_p^*, s_p^*\right) \text{ is a minimizer of } \pi_r\left(t_p, w_p, s_p\right) \\ \text{if } \pi_c\left(t_p, w_p, s_p\right) = \text{constant}. \end{cases}$$

.

Proof. See Appendix A.

Theorem 1: The interval-valued optimization problem (24) reduces to the following optimization problem

$$Maximize\ \pi_c\left(t_p, w_p, s_p\right)$$

$$\text{subject to } t_p > 0, w_p > 0, s_p > 0. \tag{25}$$

$$\text{where } \pi_c\left(t_p, w_p, s_p\right) = \frac{1}{2}\left\{\pi_L\left(t_p, w_p, s_p\right) + \pi_U\left(t_p, w_p, s_p\right)\right\} \tag{26}$$

Proof. See Appendix A.

4.1. Different meta-heuristic algorithms

The optimization problem (25) corresponding to the proposed production inventory model is highly nonlinear in nature, and its optimal solution cannot be obtained by analytical optimization techniques (gradient based technique). Therefore, to solve such real-life highly nonlinear optimization problems, meta-heuristic techniques are generally used. Here, three variants of QPSO techniques (i.e., AQPSO, GQPSO, and WQPSO), grey-wolf optimizer algorithm (GWOA), sparrow search algorithm (SSA) and teaching–learning-based optimizer algorithm (TLBOA) are used to obtain the optimal solutions of the optimization problem (25).

- Adaptive quantum-behaved particle swarm optimization (AQPSO) (Xu and Sun, (2005)).
- Gaussian quantum-behaved particle swarm optimization (GQPSO) (Coelho, (2010)).
- Weighted quantum-behaved particle swarm optimization (WQPSO) (Xi et al., (2008)).
- Grey-wolf optimizer algorithm (GWOA) (Mirjalili et al. (2014)).
- Sparrow search algorithm (SSA) (Xue and Shen (2020)).
- Teaching-learning-based optimizer algorithm (TLBOA) (Rao (2016)).

4.1.1. Motivation of using meta-heuristic algorithm

The centre of the objective function $\pi_c\left(t_p, w_p, s_p\right)$ of the optimization problem (24) is highly non-linear with respect to three decision variables, viz. selling price $\left(s_p\right)$, warranty period $\left(w_p\right)$ of the items, and production period $\left(t_p\right)$ which leads a complicacy to find analytic solution of (24). In fact, it is likely to be impossible to get an analytic solution of the problem (24).

Also, in the existing literature, a number of works in the area of production inventory have been accomplished using different variants of QPSO technique as solution methodology. Bhunia et al. (2017) first used a soft computing technique (PSO-Co algorithm) to optimize the interval-valued objective function of a production inventory model with interval valued cost components. Further, Mondal et al. (2019), Rahman et al. (2020), and Ruidas et al. (2021) used different variants of QPSO techniques to analyse optimality conditions of inventory models in interval environment.

The details of these algorithms are available in Appendix A. Also, the detailed solution methodology for optimization of an interval-valued objective function using different variants of QPSO techniques of the production inventory model with an interval-valued objective function (i.e., interval-valued average profit/average cost) is available in the works of Mondal et al. (2019), Rahman et al. (2020), and Ruidas et al. (2021).

4.1.2. Steps of the c-r optimization procedure in different variants of QPSO algorithms

For the proposed maximization problem (24), the optimal solution can be obtained by the following algorithm:

Step 1: Input the known inventory parameters.

Step 2: Find the expressions of $\pi_c\left(t_p, w_p, s_p\right)$ and $\pi_r\left(t_p, w_p, s_p\right)$ for the objective function of the maximization problem (24).

Step 3: Check whether $\pi_c\left(t_p, w_p, s_p\right)$ is constant or not.

Step 4: If $\pi_c\left(t_p, w_p, s_p\right)$ is not constant, then go to Step 5. Otherwise, go to Step 6.

Step 5: Find the optimal solution $t_p = t_p^*, w_p = w_p^*, s_p = s_p^*$ using AQPSO, GQPSO, WQPSO and MATHEMATICA maximizing $\pi_c\left(t_p, w_p, s_p\right)$ subject to $s_p > 0,\ t_p > 0, w_p > 0$.

Step 6: Find the optimal solution $t_p = t_p^*, w_p = w_p^*, s_p = s_p^*$ using AQPSO, GQPSO, WQPSO and MATHEMATICA minimizing $\pi_r\left(t_p, w_p, s_p\right)$ subject to $s_p > 0,\ t_p > 0, w_p > 0$.

Step 7: Print $t_p^*, w_p^*, s_p^*, \pi_c\left(t_p^*, w_p^*, s_p^*\right)$,

$$\pi_r\left(t_p^*, w_p^*, s_p^*\right), \pi_L\left(t_p^*, w_p^*, s_p^*\right) \text{ and } \pi_U\left(t_p^*, w_p^*, s_p^*\right).$$

Step 8: Stop.

5. Numerical illustration

To illustrate and also to validate the proposed model, two numerical examples (Example 1 accommodated with hypothetical data and Example 2 for a real case study) are considered as follows:

Example 1. Let a manufacturing industry start production with a rate that lies in the interval $[1200, 1500]$ of units/year and continues up to a certain period $\left(t_p\right)$. It is observed that the production system produces defective products at the rate that lies in the interval $[0.05, 0.06]$. The fixed demand rate of the product is in $[320, 350]$ units/year while variable demand sensitive parameters are $[1.5, 1.7]$ and $[3.5,\ 4]$ respectively. Also, the fixed unit production cost is given by $\$[20, 25]$/unit. Additionally, manufacturer provides warranty and a free service period to their customers and the warranty cost parameter is $\$[0.3, 0.5]$/unit and the holding cost is $\$[0.5, 0.7]$/unit/year and the total setup cost per cycle is $\$[500, 555]$/order. To control greenhouse gas emissions of CO_2 during production, the manufacturing company invests a fixed amount $\$[45, 55]$ and the value of variable carbon emissions sensitive parameter is $\$[22,\ 28]$. The objective of the example is to determine the best-found (or optimal) values of the production period $\left(t_p\right)$, warranty period $\left(w_p\right)$, and selling price $\left(s_p\right)$ by maximizing the interval-valued average profit of the system.

Solution. In this example, the system parameters are given by

$$[\theta_{rL},\theta_{rU}]=[0.05,0.06],[\alpha_L,\alpha_U]=[320,350],[\beta_L,\beta_U]=[1.5,1.7],[\gamma_L,\gamma_U]=[3.5,4]$$
$$[\lambda_L,\lambda_U]=[0.3,0.5],[c_L,c_U]=[20,25],[\omega_{1L},\omega_{1U}]=[45,55],[\omega_{2L},\omega_{2U}]=[22,28],$$
$$[P_{rL},P_{rU}]=[1200,1500],[h_L,h_U]=[0.5,0.7],[A_L,A_U]=[500,555].$$

To solve Example 1, three different variants of QPSO (i.e., AQPSO, GQPSO and WQPSO) algorithms, GWOA SSA and TLBOA are used. The best-found (optimal) results obtained from six different metaheuristic algorithms are displayed in Table 3, whereas the worst-found results are also reported in Table 4. 50 independent runs were taken for each algorithm for statistical experiments. The results obtained from the statistical experiment are shown in Table 5.

In Fig. 3, the concavity of the centre of the average profit $\left(\pi_c\right)$ is shown graphically with respect to $\left(w_p,\ s_p\right)$ by keeping t_p fixed at its

Table 3
Best-found (optimal) solution for Example 1.

Algorithms	$\langle \pi_c, \pi_r \rangle$ (in \$/year)	$[\pi_L, \pi_U]$ (in \$/year)	p (in \$/unit)	w_p (in year)	t_p (in year)	T (in year)
AQPSO	⟨3370.971758, 1500.785279⟩	[1870.186478, 4871.757037]	59.429939	0.903772	0.157790	1.773344
GQPSO	⟨3370.971758, 1500.831761⟩	[1870.139997, 4871.803519]	59.430597	0.921217	0.157790	1.773343
WQPSO	⟨3370.971758, 1500.829539⟩	[1870.142219, 4871.801297]	59.430566	0.903772	0.157790	1.773344
GWOA	⟨3370.971750, 1500.907361⟩	[1870.064389, 4871.879110]	59.433478	0.907657	0.157801	1.773540
SSA	⟨3370.971758, 1500.831995⟩	[1870.139763, 4871.803753]	59.430600	0.903776	0.157790	1.773343
TLBOA	⟨3370.971758, 1500.831910⟩	[1870.139847, 4871.803668]	59.430599	0.903773	0.157790	1.773343

Table 4
Worst found results obtained by different algorithms for Example 1.

Algorithms	$\langle \pi_c, \pi_r \rangle$ (in \$/year)	$[\pi_L, \pi_U]$ (in \$/year)	p (in \$/unit)	w_p (in year)	t_p (in year)	T (in year)
AQPSO	⟨3370.971757, 1500.715681⟩	[1870.256077, 4871.687438]	59.428974	0.903772	0.157790	1.773345
GQPSO	⟨3370.971758, 1500.831848⟩	[1870.139909, 4871.803606]	59.430599	0.855972	0.157790	1.773343
WQPSO	⟨3370.971716, 1502.206030⟩	[1868.765686, 4873.177746]	59.449549	0.903774	0.157790	1.773301
GWOA	⟨3370.971340, 1498.813353⟩	[1872.157987, 4869.784693]	59.416453	0.846291	0.157825	1.774351
SSA	⟨3370.971758, 1500.832042⟩	[1870.139715, 4871.803800]	59.430601	0.903777	0.157790	1.773343
TLBOA	⟨3370.971758, 1500.831821⟩	[1870.139937, 4871.803578]	59.430598	0.903770	0.157790	1.773343

Table 5
Analysis of statistical significance of different algorithms for Example 1.

Algorithms	Best found π_c (in \$)	Worst found π_c (in \$)	Mean of π_c(in \$)	Median of π_c (in \$)	Standard Deviation
AQPSO	**3370.971758**	3370.971716	3370.971755	**3370.971758**	$\mathbf{2.71293 \times 10^{-7}}$
GQPSO	**3370.971758**	**3370.971758**	**3370.971758**	**3370.971758**	**0**
WQPSO	**3370.971758**	3370.971716	3370.971755	**3370.971758**	$\mathbf{9.95791 \times 10^{-6}}$
GWOA	**3370.971758**	3370.971723	3370.971658	3370.971676	$\mathbf{7.34047 \times 10^{-11}}$
SSA	**3370.971758**	**3370.971758**	**3370.971758**	**3370.971758**	**0**
TLBOA	**3370.971758**	**3370.971758**	**3370.971758**	**3370.971758**	**0**

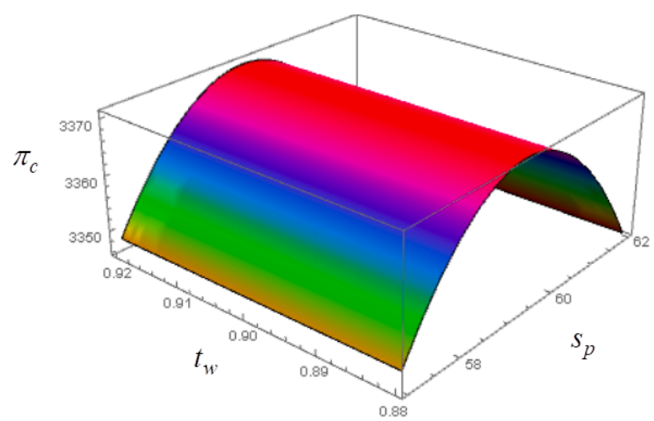

Fig. 3. Concavity of the centre of the average profit with respect to w_p and s_p of Example 1.

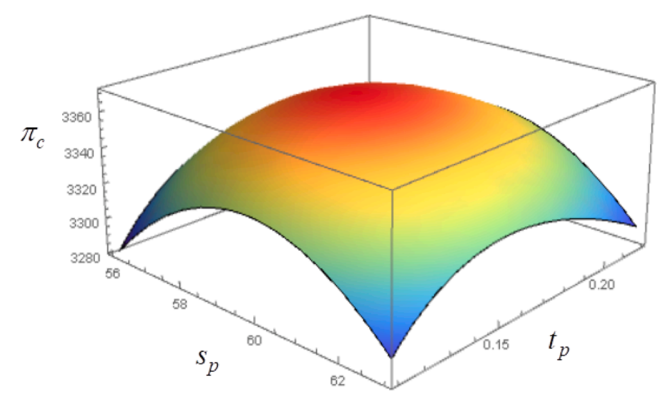

Fig. 4. Concavity of the centre of the average profit with respect to t_p and s_p of Example 1.

optimal value. Similarly, the concavity of the centre of the average profit (π_c) with respect to (t_p, s_p) is shown in Fig. 4 and with respect to (t_p, w_p) is shown in Fig. 5. Fig. 6 shows the changes of the interval-valued inventory level with respect to time.

5.1. Analysis of variance (ANOVA)

From the solutions of Example 1, it can be noticed that AQPSO, GQPSO, WQPSO, SSA and TLBOA give better performance in comparison with GWOA. However, from the statistical experiment (Table 5), it is seen that the standard deviation in SSA is minimum. To determine the significance of runs obtained from GQPSO and five algorithms, one of the statistical tests i.e., analysis of variance (ANOVA) is executed for Example 1. In this test, SSA is taken as the controlling algorithm and the results obtained from ANOVA test are displayed in Table 6. This experiment is performed in Microsoft Excel 2019. In Table 6, the values of sum of squares (*SS*), degree of freedom (*df*) and mean sum of squares (*MS*) between groups and within group, calculated *F*-statistic values (*F*), *F*-critical values (*F-crit*) and *P*-values are reported.

From Table 6, it can be observed that *F*-static values for all algorithms are greater than *F*-critical value. Therefore, the null hypothesis is rejected. Again, the *P-values* (**bold faced**) of GQPSO versus AQPSO and TLBOA are less than 0.05. Therefore GQPSO performs significant in compare to AQPSO and TLBOA at 5 % significance level.

5.2. Convergence graph

Here, the convergence rates of different algorithms (viz. AQPSO, GQPSO, WQPSO, GWOA, SSA and TLBOA) to the best-found values of centre of the average profit of Example 1 are shown in Fig. 7.

From Fig. 7, it can be concluded that GWOA has the lowest

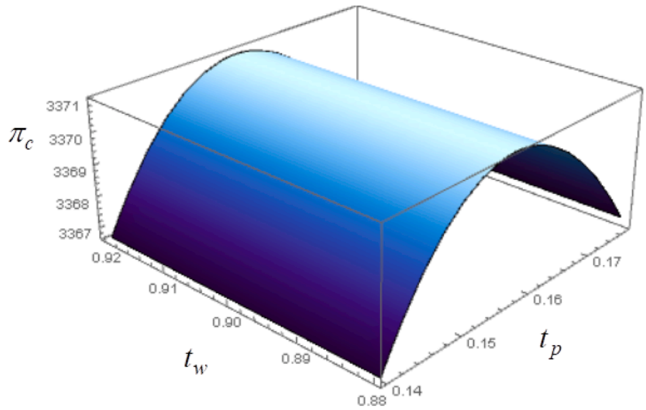

Fig. 5. Concavity of the centre of the average profit with respect to t_p and w_p of Example 1.

convergence rate to the best-found solution whereas all other five algorithms have the same rates of convergence.

Observations and discussions

(i) From Table 3, it can be observed that the best-found values of the centre of the average profit obtained by AQPSO, GQPSO, WQPSO, SSA and TLBOA techniques be the same up to six places of decimals and it is different from GWOA. Therefore, the efficiency of GWOA in solving Example 1 is less in compare to the other five algorithms. Moreover, the best-found and worst found values of the centre of avergae profit for Example 1 (cf. Table 5) be the same with those obtained from GQPSO, SSA and TLBOA.

(ii) From ANOVA test (cf. Table 6), it is noticed that GQPSO appears to be the most efficent algorithm in solving Example 1.

(iii) Also, the concavities of the average profit, or the centre of average profit and bounds of average profits (Figs. 3–5) prove the existence of optimality of the obtained solutions.

(iv) For almost all the production-based companies, the adoption of emissions controlling technology is mandatory, given the current environmental situation. For our proposed production inventory problem, the cost of emissions controlling investment is considered as a linear time-dependent function. Again, for the numerical Example 1, the changes in a carbon emissions-controlled investment and the centre of average profit with regard to the production period of the manufacturing firm are simultaneously shown graphically in Fig. 8. From this figure, it can be observed that both the bounds of carbon emissions controlling investment increase strictly as the production period of the firm increases. It can be seen that, after a long duration of a production period, this investment is even greater than the centre of the average profit of the system. Thus, the emissions-controlled investment for the proposed problem has a significant impact on the optimal policy of the manufacturing firm.

5.3. Practical Example

Example 2. This example is based on the market study of the water purifier, which is produced by a local manufacturing company in Kolkata, West Bengal, India. In this example, US dollar ($) is used as currency, however, the survey was done using Indian rupee (₹) as currency. The manufacturer produces water purifier approximately $[P_{rL}, P_{rU}] = [1200, 1500]$ pieces per year. A part of the produced water purifier was found to be defective, and the defective rate is $[\theta_{rL}, \theta_{rU}] = [0.04, 0.06]$ of the whole production. The fixed demand rate of the water purifier is

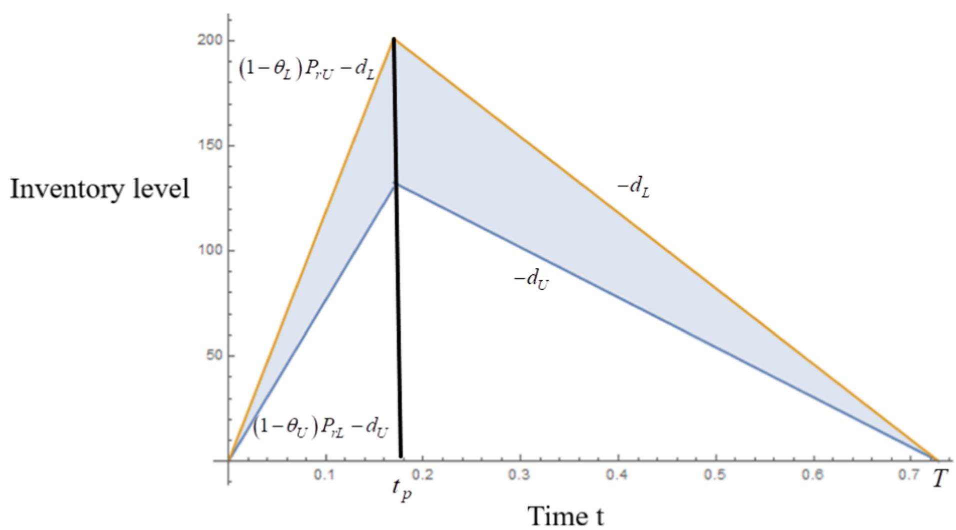

Fig. 6. Changes of inventory level with respect to time for Example 1.

Table 6
Results of ANOVA test corresponding to Example 1.

GQPSO Vs	Count	Average	Variance	Source of Variation						F	F-crit	P-value
				Between Groups			Within Group					
				SS	Df	MS	SS	df	MS			
AQPSO	50	3370.971758	7.51×10^{-14}	1.6×10^{-13}	1	1.6×10^{-13}	3.68×10^{-12}	98	3.75×10^{-14}	4.26087	3.9381	0.041642
WQPSO	50	3370.971755	1.01×10^{-10}	1.69×10^{-10}	1	1.69×10^{-10}	4.96×10^{-9}	98	5.06×10^{-11}	3.34046	3.9381	0.070639
GWOA	50	3370.971658	5.5×10^{-9}	2.51×10^{-7}	1	2.51×10^{-7}	2.69×10^{-7}	98	2.75×10^{-9}	91.48507	3.9381	1
SSA	50	3370.971758	0	4.05×10^{-21}	1	4.05×10^{-21}	0	98	0	65, 535	3.9381	1
TLBOA	50	3370.971758	0	$3.17 \times 10–20$	1	3.17×10^{-20}	$1.32 \times 10–21$	98	1.35×10^{-23}	65, 535	3.9381	$2.59 \times 10–70$

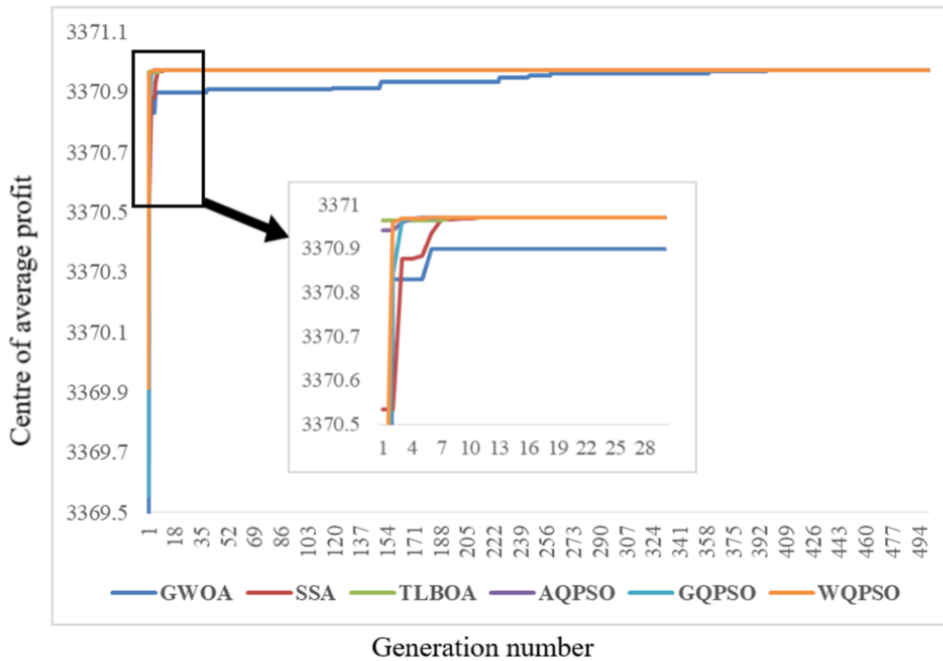

Fig. 7. Convergence rate of various algorithms for Example 1.

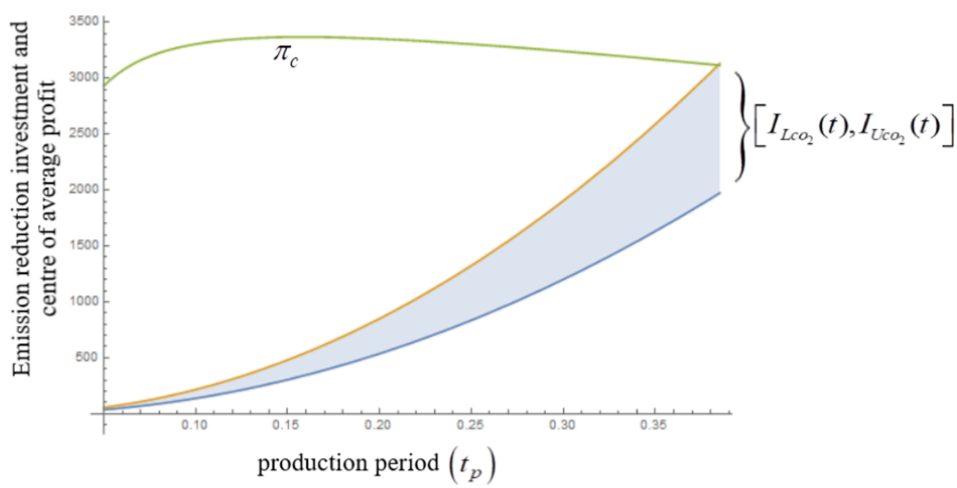

Fig. 8. Centre of the average profit and disposal investment rate of hazardous waste with respect to cycle length for Example 1.

Table 7
Best-found (or optimal) solution for Example 2.

Algorithms	$\langle \pi_c, \pi_r \rangle$ (in \$/year)	$[\pi_L, \pi_U]$ (in \$/year)	p (in \$/unit)	w_p (in year)	t_p (in year)	T (in year)
AQPSO	⟨6309.311275, 4252.599906⟩	[2056.711369, 10561.911181]	132.714490	3.775475	0.135188	1.574532
GQPSO	⟨6309.311947, 4258.683503⟩	[2050.628444, 10567.995450]	132.779704	3.794595	0.135063	1.573293
WQPSO	⟨6309.311947, 4258.683876⟩	[2050.628071, 10567.995823]	132.779707	3.794595	0.135063	1.573293
GWOA	⟨6309.311275, 4252.599906⟩	[2056.711369, 10561.911181]	132.714490	3.775475	0.135188	1.574532
SSA	⟨6309.311947, 4258.682813⟩	[2964.145462, 9230.670342]	132.779698	3.794595	0.135063	1.573293
TLBOA	⟨6309.311947, 4258.683773⟩	[2050.628174, 10567.995720]	132.779706	3.794598	0.135063	1.573293

$[320, 350]$ units/year while variable demand sensitive parameters are $[\beta_L, \beta_U] = [5, 6]$ and $[\gamma_L, \gamma_U] = [1.3, 2.4]$ respectively. Also, the fixed unit production cost of water purifier is $[c_L, c_U] = \$[50, 60]$/unit. Additionally, the holding cost of water purifier is $[h_L, h_U] = \$[1.5, 1.6]$/unit/year and the total setup cost per cycle is $[A_L, A_U] = \$[500, 520]$/order. The manufacturing company offers a free servicing cost to the customers for repairing during the warranty period of the product and the servicing cost is dependent on the warranty period of the cost. The average free servicing cost per water purifier is $\$[2.5, 3]w_p$. To control greenhouse gas emissions of CO_2 during production, the manufacturing company invests a fixed amount $\$[45, 55]$ and the value of variable carbon emissions sensitive parameter is $\$[22, 28]$. All data are collected from a reliable source by the survey of the market in Kolkata, West Bengal, India. Finally, the managers of the warranty period manufacturing company wanted to determine the optimal selling price, production period and warranty period that maximize the average profit. Assume that 1\$ = ₹80.

Solution: According to our proposed technique to solve interval-

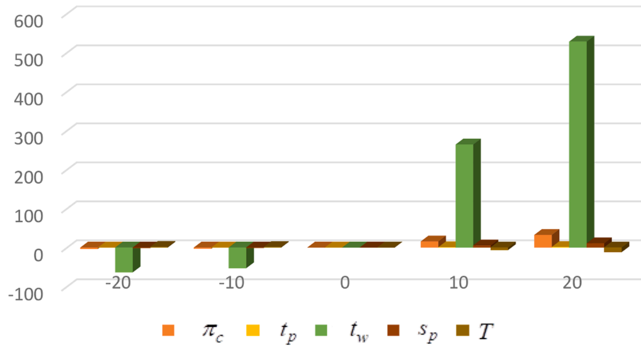

Fig. 9. Impact of $[\alpha_L, \alpha_U]$ in the optimal policy for Example 1.

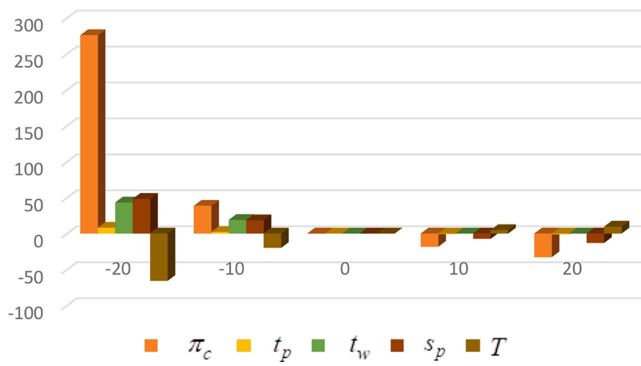

Fig. 10. Impact of $[\gamma_L, \gamma_U]$ in the optimal policy for Example 1.

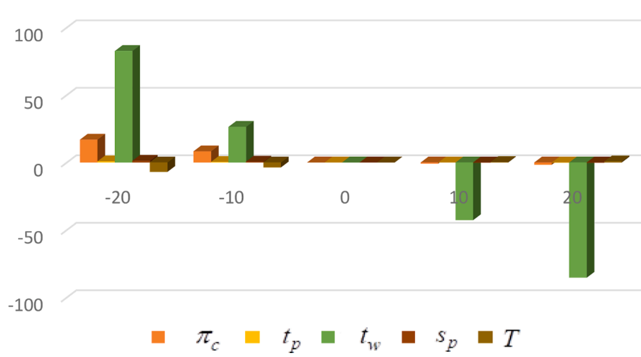

Fig. 11. Impact of $[c_L, c_U]$ in the optimal policy for Example 1.

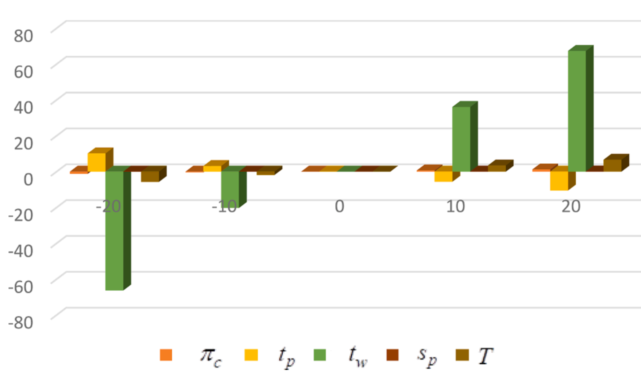

Fig. 12. Impact of $[P_{rL}, P_{rU}]$ in the optimal policy for Example 1.

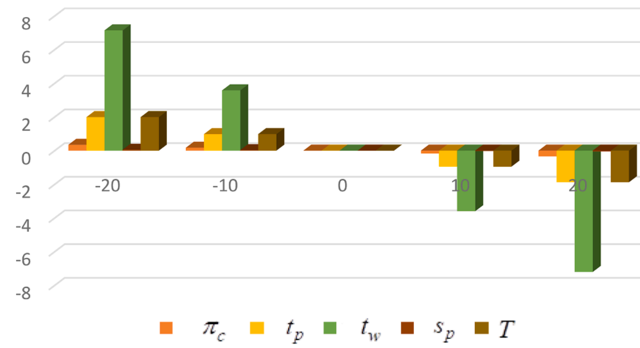

Fig. 13. Impact of $[h_{cL}, h_{cU}]$ in the optimal policy for Example 1.

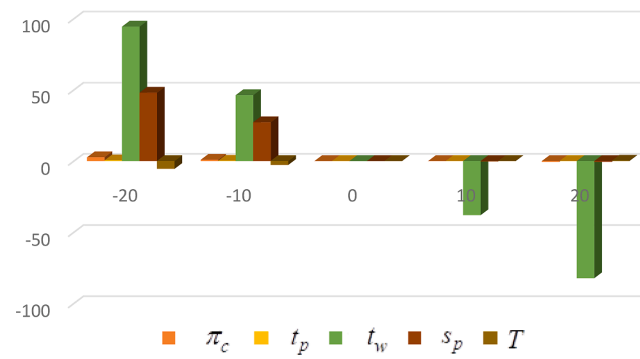

Fig. 14. Impact of $[\mu_L, \mu_U]$ in the optimal policy for Example 1.

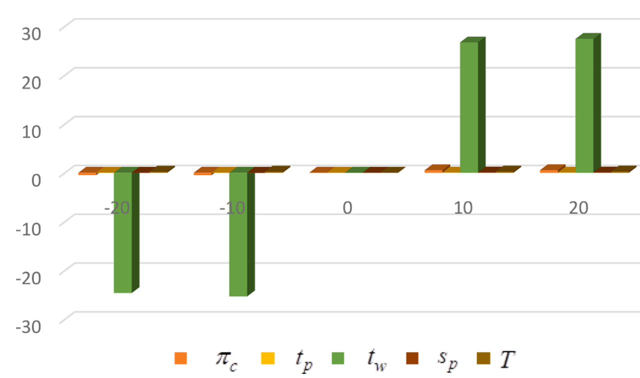

Fig. 15. Impact of $[\varpi_{1L}, \varpi_{1U}]$ in the optimal policy for Example 1.

valued optimization problems, both the possible optimal average profit of the company and the optimal results are summarized in Table 7.

6. Sensitivity analysis

To show the impact of various known inventory parameters on the centre of the average profit (π_c), production period (t_p), warranty period (w_p) and selling price (s_p), the sensitivity analyses are carried out using Example 1 by changing both the bounds of the interval-valued parameters from -20% to 20%. The results of this sensitivity analysis are shown graphically in Figs. 9–15.

From Figs. 9–15, the followings observations are made.

(i) The centre of the average profit (π_c) of Example 1 is largely sensitive with respect to $[\alpha_L, \alpha_U]$ and $[\gamma_L, \gamma_U]$. Again, π_c is moderately sensitive directly regarding $[c_L, c_U]$. On the other hand, π_c is insensitive with respect to $[P_{rL}, P_{rU}]$, $[h_{cL}, h_{cU}]$, $[\lambda_L, \lambda_U]$,$[\omega_{1L}, \omega_{1U}]$ and $[h_{cL}, h_{cU}]$.

(ii) The business period (T) of Example 1 is equally reverse sensitive with respect to $[\lambda_L, \lambda_U]$,$[\gamma_L, \gamma_U]$ whereas it is less sensitive directly with respect to $[\alpha_L, \alpha_U]$. Further, T is insensitive regarding $[P_{rL}, P_{rU}]$, $[h_{cL}, h_{cU}]$, $[\omega_{1L}, \omega_{1U}]$ and $[c_L, c_U]$.
(iii) The production period (t_p) of Example 1 is moderately sensitive directly with respect to $[\alpha_L, \alpha_U]$,$[\gamma_L, \gamma_U]$, whereas it is insensitive with respect to the rest of the parameters.
(iv) The warranty period (w_p) of Example 1 is highly sensitive with respect to $[\alpha_L, \alpha_U]$, $[c_L, c_U]$, $[P_{rL}, P_{rU}]$, $[\lambda_L, \lambda_U]$, $[\gamma_L, \gamma_U]$ and $[\omega_{1L}, \omega_{1U}]$. Further, it is insensitive with respect to $[h_{cL}, h_{cU}]$.
(v) The selling price (s_p) of Example 1 is highly sensitive reversely with respect to $[\lambda_L, \lambda_U]$, $[\gamma_L, \gamma_U]$ while it is insensitive with respect to the rest of the system parameters.

7. Managerial insights

From the sensitivity analysis, the following observations may be suggested to the managers and decision makers of business firms:

(i) From the sensitivity analysis, it is observed that the centre of the average profit of the system increases most significantly as the initial demand rate of the product increases. It is a natural phenomenon and it holds in the proposed model while the average profit of the system decreases as the selling price of the products gives a negative impression to customers. Therefore, the authority of the manufacturing firm should care about this.
(ii) Controlling carbon emissions in the environment is a challenging task to industrial managers. Government has imposed different rules and regulations regarding carbon emissions in the environment. Also, decision makers have a responsibility to control carbon emissions from their industry to the environment. Therefore, decision makers must think about the investment in carbon emissions for the long run of their production process for emitting less carbon to the environment and save the environment as well as human civilization.
(iii) Warranties of products have a big influence on customer demand. The demand of the product is increased after considering the warranty concept. Therefore, decision makers should think about the warranty policies of products in order to increase the demand of the product, as well as to increase their profit.
(iv) Selling price of the product is also another important issue to increase the demand of an item. If the selling price of an item is increased, then customers are unable to buy that product. Therefore, decision makers must also think about this matter in order to increase the demand of their product.

8. Conclusions

In the current competitive market, each manufacturing company offers various attractive policies to attract more customers. In the proposed model, the warranty of products is considered because various companies commonly apply it to their manufactured products. Again, demand depends on the selling price of the product and the warranty period. Considering the ever-growing requirements of customers in different sectors of the electronic gadgets market and observing, in parallel, the current competitive situation in marketing, products with a warranty period are more popular among consumers because these products are more reliable and perform better. Furthermore, correctly setting both the selling price of a product and the production period is a complex process. Additionally, in order to reduce the risk of polluting the environment, every manufacturer needs to invest a certain amount of money to reduce carbon emissions in the production process. This investment is taken as linearly time-dependent. Also, taking into account the flexible behaviour of different inventory parameters, this model has been developed in interval environment that makes its assumptions more realistic. However, to consider the flexibility of different parameters involved in the system, the optimization problem corresponding to the model appears to be interval valued and it is solved using the c-r optimization method. Further, because of the complex behaviour of the centre of the average profit of the system with respect to different decision variables, the average profit cannot be optimized analytically and the implementation of different meta-heuristic algorithms becomes a necessity.

This research may be helpful to the authorities and practitioners of manufacturing firms who plan the production processes in order to estimate optimal policies. This model presents a rough idea to the manufacturers about the optimal policies to take in conferring warranties on their products and in reducing carbon emissions during production. Thus, the main advantage of this model is that it accommodates a balanced relationship between a production rate, emissions reduction levels of the firm, and warranty periods of the produced products. This model can be implemented to make optimal decisions regarding the manufacturing and marketing of cell phones, laptop computers, desktop computers, televisions, induction ovens, LED lights, capacitors, or any type of electronic product.

Although in this work, different ideas were applied during the formulation of the model, such as the warranty policy and the emissions reduction effort, as well as the warranty period's being dependent upon production costs, we still believe there are some limitations in our proposed model. First, in this model, the deterministic rate of interval-valued production is considered. In reality, however, production rates are dependent on time or various other factors (viz. demand, supply rate of raw materials, and transportation costs). Second, there is no theoretical proof of the optimal policy of the proposed model.

Therefore, one can extend this model by considering a time-dependent production rate or by considering the shortage of a given product. Nowadays, green credentials have become widely popular. Given this, one can consider the “green level” of a product and its effect on customer demand during the model formulation. One can also enrich this model through yet another dimension of a business strategy by introducing a trade credit facility, an advanced payment facility, or an applied discount facility. In the future, one can study these issues and formulate models with more optimal decisions.

CRediT authorship contribution statement

Amalesh Kumar Manna: Conceptualization, Investigation, Methodology, Validation, Writing - original draft. **Subhajit Das:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Visualization, Writing - original draft. **Ali Akbar Shaikh:** Supervision, Formal analysis, Investigation. **Asoke Kumar Bhunia:** Validation, Writing – review & editing, Supervision. **Ilkyeong Moon:** Validation, Writing – review & editing, Supervision.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A

Algebra of intervals and interval valued functions: Let K_c be the set of all nonempty compact intervals of $\mathbb{R}$, i.e., $K_c = \{[a_L, a_U] : a_L, a_U \in \mathbb{R} \text{ and } a_L \leqslant a_U\}$.

Now the parametric forms of an element $[a_L, a_U] \in K_c$ are defined as follows:

(i) Increasing parametric form (IPF): $[a_L, a_U] = \{a(\lambda) = a_L + \lambda(a_U - a_L) : \lambda \in [0,1]\}$

(ii) Decreasing parametric form (DPF): $[a_L, a_U] = \{a(\lambda) = a_U + \lambda(a_L - a_U) : \lambda \in [0,1]\}$.

Therefore, the set of all compact intervals in parametric form is denoted by K_p and it is defined by

$$K_P = \{a(\lambda) : a(\lambda) \text{ is IPF (or DPF) of } [a_L, a_U], \forall \lambda \in [0,1] \text{ and } \forall [a_L, a_U] \in K_c\}$$

Clearly, the sets K_c and K_p are equivalent.

Definition A.1. Let $I_1 = \{a(\lambda) : \lambda \in [0,1]\}, I_2 = \{b(\lambda) : \lambda \in [0,1]\} \in K_p$ and let $\mu \in \mathbb{R}$. Different arithmetic operations on K_p are defined as follows:

(i) Addition: $I_1 + I_2 = \{a(\lambda_1) + b(\lambda_2) : \lambda_1, \lambda_2 \in [0,1]\}$
(ii) Subtraction: $I_1 - I_2 = \{a(\lambda_1) - b(\lambda_2) : \lambda_1, \lambda_2 \in [0,1]\}$
(iii) Multiplication: $I_1 I_2 = \{a(\lambda_1) b(\lambda_2) : \lambda_1, \lambda_2 \in [0,1]\}$
(iv) Division: $I_1 / I_2 = \left\{\frac{a(\lambda_1)}{b(\lambda_2)} : \lambda_1, \lambda_2 \in [0,1]\right\}$, provided $0 \notin I_2$.
(v) Parametric difference: $I_1 -_p I_2 = \{a(\lambda) - b(\lambda) : \lambda \in [0,1]\}$
(vi) Scalar Multiplication: $\mu \odot I_1 = \{\mu a(\lambda) : \mu \in [0,1]\}$

(vii) Equality: $I_1 = I_2$ if and only if $a(\lambda) = b(\lambda), \forall \lambda \in [0,1]$.

Definition A.2. Let an interval valued function $F : D \subseteq \mathbb{R}^n \to K_c$ be defined as $G(t) = [G_L(t), G_U(t)]$, where $G_L, G_U : D \subseteq \mathbb{R}^n \to \mathbb{R}$ with $G_L(t) \leqslant G_U(t)$, $\forall t \in D$.

The parameterized form (IPF) or p-interval valued function of $G(t)$ is defined as $G : D \subseteq \mathbb{R}^n \to K_p$ and it is defined by $G(t) = \left\{\widetilde{G}(t, \lambda) = G_L(t) + \lambda(G_U(t) - G_L(t)) : \lambda \in [0,1]\right\}, \forall t \in D$.

Definition A.3. The function $G(t) = [G_L(t), G_U(t)]$ is said to be p-differentiable at t_0 if $\lim_{h \to \theta_{\mathbb{R}^n}} \frac{G(t_0+h) -_p G(t_0)}{h}$ exists finitely. The p-derivative of G at t_0 is denoted by $G'(t_0)$.

Definition A.4. Let $Y : [a,b] \to K_c$ be a p-differentiable interval valued function of real single variable and let $F : [a,b] \times K_c \to K_c$ be a continuous interval valued function. Then an interval differential equation is defined as follows:

$$\frac{dY}{dt} = F(t, Y), \quad t \in [a,b]$$

$$\text{where } Y(t) = [Y_L(t), Y_U(t)] \text{ and } F(t, Y) = [F_L(t, Y), F_U(t, Y)]. \tag{A.1}$$

Now the parametric representation of (A.1) is given below:

$$\frac{d\widetilde{Y}(t,\lambda_1)}{dt} = \widetilde{F}\left(t, \widetilde{Y}(t,\lambda_1), \lambda_2\right), \quad t \in [a,b] \text{ and } \lambda_1, \lambda_2 \in [0,1] \text{ where } Y(t) = \left\{\widetilde{Y}(t,\lambda_1) = Y_L(t) + \lambda_1(Y_U(t) - Y_L(t)) : \lambda_1 \in [0,1]\right\} \text{ and } F(t, Y)$$
$$= \left\{\widetilde{F}\left(t, \widetilde{Y}(t,\lambda_1), \lambda_2\right) = F_L\left(t, \widetilde{Y}(t,\lambda_1)\right) + \lambda_2\left(F_U\left(t, \widetilde{Y}(t,\lambda_1)\right) - F_L\left(t, \widetilde{Y}(t,\lambda_1)\right)\right) : \lambda_1, \lambda_2 \in [0,1]\right\}. \tag{A.2}$$

Proposition A.1: The equations (A.1) and (A.2) are equivalent.

Appendix B

Proof of Proposition 1. Let $t_p = t_p^*$, $w_p = w_p^*$ and $s_p = s_p^*$ be the global maximal point of $\pi_c(t_p, w_p, s_p)$ if and only if

$$\pi_c\left(t_p^*, w_p^*, s_p^*\right) \geqslant_{\max} \pi_c(t_p, w_p, s_p)$$

$$\Leftrightarrow \begin{cases} \pi_c\left(t_p^*, w_p^*, s_p^*\right) \rangle \pi_c(t_p, w_p, s_p) \text{ if } \pi_c\left(t_p^*, w_p^*, s_p^*\right) \neq \pi_c(t_p, w_p, s_p), \\ \qquad \forall s_p > 0,\ 0 < t_p, w_p < T \text{ and } (t_p, w_p, s_p) \neq \left(t_p^*, w_p^*, s_p^*\right) \\ \pi_r\left(t_p^*, w_p^*, s_p^*\right) \leqslant \pi_r(t_p, w_p, s_p) \text{ if } \pi_c\left(t_p^*, w_p^*, s_p^*\right) = \pi_c(t_p, w_p, s_p),\ \forall s_p > 0,\ 0 < t_p, w_p < T. \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi_c\left(t_p^*, w_p^*, s_p^*\right) \rangle \pi_c\left(t_p, w_p, s_p\right) \text{ if } \pi_c\left(t_p, w_p, s_p\right) \neq \text{Constant} \\ \pi_r\left(t_p^*, w_p^*, s_p^*\right) \leqslant \pi_r\left(t_p, w_p, s_p\right) \text{ if } \pi_c\left(t_p, w_p, s_p\right) = \text{Constant} \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi_c\left(t_p, w_p, s_p\right) \text{ has a maximizer at } t_p = t_p^*,\ w_p = w_p^* \text{ and } s_p = s_p^* \text{ if } \pi_c\left(t_p, w_p, s_p\right) \neq \text{constant} \\ \pi_r\left(t_p, w_p, s_p\right) \text{ has a minimizer at } t_p = t_p^*,\ w_p = w_p^* \text{ and } s_p = s_p^* \text{ if } \pi_c\left(t_p, w_p, s_p\right) = \text{constant.} \end{cases}$$

This completes the proof.

Proof of Theorem 1. Since the centre of average profit function $\pi_c\left(t_p, w_p, s_p\right)$ is dependent on the variables $\left(t_p, w_p, s_p\right)$, therefore, from Proposition 1, interval-valued optimization problem (23) is equivalent to the optimization problem which optimizes the centre of average profit i.e., $\pi_c\left(t_p, w_p, s_p\right)$.

Details about particle swarm optimization (PSO) and quantum behaved particle swarm optimization (QPSO)

Particle swarm optimization technique is one of the most popular algorithms proposed by Eberhart and Kennedy (1995) for solving different types of real-life nonlinear optimization problems. This algorithm was developed from the inspiration of natural behaviour of bird flocking or fish schooling. In PSO algorithm, position of the particles' (i.e., potential solutions) move throughout the search space. Initially, a random particles' positions (set of potential solutions) vector $\widetilde{x}_i = \left(x_{i,1}, x_{i,2}, ..., x_{i,n}\right) \in \mathbb{R}^n$ is initialized and after those the best-found particles' positions (optimal solutions) be searched in every iteration. In each iteration, every particle's position $\widetilde{x}_i$ is updated with the help of two best positions: The first best position is called **personal best position and** it is denoted by p_i whereas the other swarm's best position is named as **global best position** and it is denoted by g.

Let us assume that 'n' be the number of decision variables of the optimization problem and the number of individuals in population as p_size. In PSO algorithm at k- th iteration $(k = 1, 2, ..., k_{\max})$, i- th particle $(1 \leqslant i \leqslant p\_size)$ has the following attributes:

(i) $\widetilde{x}_i^{(k)} = \left(x_{i,1}^{(k)}, x_{i,2}^{(k)}, ..., x_{i,n}^{(k)}\right)$ be the position of i- th particle at k- th iteration in the search space.
(ii) $p_i^{(k)} = \left(p_{i,1}^{(k)}, p_{i,2}^{(k)}, ..., p_{i,n}^{(k)}\right)$ be the personal best position of i- th particle at k- th iteration.
(iii) $g^{(k)} = \left(g_1^{(k)}, g_2^{(k)}, ..., g_n^{(k)}\right)$ be the global best position of all swarm's particles at k- th iteration.

The personal best position of each particle is calculated as follows:

$$p_i^{(0)} = \widetilde{x}_i^{(0)} \quad \text{and} \quad p_i^{(k+1)} = \begin{cases} p_i^{(k)}, \quad if\ f(p_i^{(k)}) \geq^{\max} f(\widetilde{x}_i^{(k+1)}) x_i^{(k+1)}, \quad if\ f(p_i^{(k)}) <^{\min} f(\widetilde{x}_i^{(k+1)}) \end{cases}$$

where $f(u) = \left[f_L(u), f_U(u)\right] = \langle f_c(u), f_r(u) \rangle$ be the interval valued objective function; the inequality signs $\geqslant^{\max}$ and $\leqslant^{\min}$ depend on the definitions of Bhunia and Samanta (2014).

The global best position found by any particle during all previous iterations is defined as follows:

$$g^{(k+1)} = \arg \max_{p_i} f_{c/r}(p_i^{(k+1)}), \quad 1 \leqslant i \leqslant p\_size \tag{B.6}$$

$$\text{where } f_{c/r}\left(p_i^{(k+1)}\right) = \begin{cases} f_c\left(p_i^{(k+1)}\right), \text{ if } f_c\left(p_i^{(k+1)}\right) > f_c\left(x_i^{(k+1)}\right) \\ f_r\left(p_i^{(k+1)}\right), \text{ if } f_c\left(p_i^{(k+1)}\right) = f_c\left(x_i^{(k+1)}\right) \\ f_c\left(x_i^{(k+1)}\right), \text{ if } f_c\left(p_i^{(k+1)}\right) \langle f_c\left(x_i^{(k+1)}\right) \end{cases}$$

Different improved version of PSO are available in the existing literature. Among them, quantum-behaved particle swarm optimization (QPSO) is one of the most improved versions of algorithms for solving optimization problem.

Quantum behaved particle swarm optimization (QPSO) is one of the most popular swarm intelligent based efficient algorithms. The behaviour of the particle follows quantum mechanics of the particle. In quantum mechanics, position and velocity of a particle cannot be determined simultaneously due to uncertainty principle. Keeping in mind about this concept, Sun et al. (2004) first introduced improved PSO algorithms and named as QPSO algorithm. The iterative equation for the position of the particle in QPSO is given by

$$x_{i,j}^{(k+1)} = \begin{cases} \widetilde{p}_{i,j}^{(k)} + \lambda \left| m_j^{(k)} - x_{i,j}^{(k)} \right| \ln\left(\dfrac{1}{\tau_j^{(k+1)}}\right) \quad, \forall j = 1, 2, ..., n, \quad \text{if } r > 0.5 \\ \widetilde{p}_{i,j}^{(k)} - \lambda \left| m_j^{(k)} - x_{i,j}^{(k)} \right| \ln\left(\dfrac{1}{\tau_j^{(k+1)}}\right) \quad, \forall j = 1, 2, ..., n, \quad \text{if } r \leqslant 0.5 \end{cases} \tag{B.7}$$

where $\widetilde{p}_{i,j}^{(k)}$ be the components of local attractor $\widetilde{p}_i = (\widetilde{p}_{i,1}, \widetilde{p}_{i,2}, ..., \widetilde{p}_{i,n})$ of each particle and is defined as

$$\widetilde{p}_{i,j}^{(k)} = \xi_j p_{i,j}^{(k)} + (1 - \xi_j) g_j^{(k)}; \tag{B.8}$$

where ξ_j and $r \sim U(0,1)$; $\tau_j^{(k+1)} \sim U(0,1)$ at $(k+1)$-th iteration; $U(0,1)$ be the uniformly distributed random number between 0 and 1; λ be the contraction–expansion coefficient which can be tuned to control the convergence speed of the algorithm and it decreases from $\lambda_0(=1.0)$ to $\lambda_1(=0.5)$

and is computed by

$$\lambda = \lambda_0 + (\lambda_1 - \lambda_0)\frac{(k-1)}{k_{\max}}; \tag{B.9}$$

and $m^{(k)}$ be the *mainstream thought or mean best position* (i.e., mean of $p_i^{(k)}$ of all swarm's particles at k-th iteration) defined as

$$m^{(k)} = \left(m_1^{(k)}, m_2^{(k)}, \ldots, m_n^{(k)}\right)$$

$$= \left(\frac{1}{p\_size}\sum_{i=1}^{p\_size} p_{i,1}^{(k)}, \frac{1}{p\_size}\sum_{i=1}^{p\_size} p_{i,2}^{(k)}, \ldots, \frac{1}{p\_size}\sum_{i=1}^{p\_size} p_{i,n}^{(k)}\right) \tag{B.10}$$

Due to the performance of QPSO algorithm, several versions of QPSO are reported in the existing literature. In this connection, the existing improved versions of QPSO, like, Adaptive quantum-behaved particle swarm optimization (AQPSO) (Xu and Sun, 2005), weighted quantum-behaved particle swarm optimization (WQPSO) (Xi et al., 2008) and Gaussian quantum-behaved particle swarm optimization (GQPSO) (Coelho, 2010) are worth mentioning.

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