Strategic Investment to Reduce Setup Times in the Economic Lot Scheduling Problem

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This article considers the Economic Lot Scheduling Problem where setup times and costs can be reduced by an initial investment that is amortized over time. The objective is to determine a multiple-item single facility cyclic schedule to minimize the long run average holding and setup costs plus the amortized investment. We develop a lower bound on the long run average inventory carrying and setup costs as a function of the setup times, and show that this lower bound is increasing concave on the setup times when the out-of-pocket setup costs are zero or proportional to the setup times. We then develop a model that may be helpful in deciding the magnitude and the distribution of a one-time investment in reducing the setup times when the investment is amortized over time. Numerical results based on randomly generated problems, and on Bomberger's ten item problem indicate that significant overall savings are possible for highly utilized facilities. Most of the savings are due to a significant reduction in the long run average holding cost.

1. INTRODUCTION

Due to economies of scale it is common in industry to produce several items in a single facility. Typically, these facilities can only produce one item at a time, and have to be stopped and prepared, i.e., *setup*, at a cost of time and money, before the start of the production run of a different item. A scheduling problem arises because of the need to coordinate the setups and the production runs of the items without ever scheduling two tasks at the same time. The problem of scheduling the production of several items in a single facility to minimize the long run average inventory carrying and setup cost is known in the literature as the Economic Lot Scheduling Problem (ELSP). The ELSP has been widely studied for over 30 years, and it is typically assumed that production and demand rates are known item-dependent constants and that setup times and setup costs are known item-dependent, but sequence independent, constants. In addition, research in the ELSP has focused on *cyclic schedules*, i.e., schedules that are repeated periodically. Moreover, almost all researchers have restricted attention to cyclic schedules that satisfy the Zero Switch Rule (ZSR). This rule states that a production run for any particular item can be started only if its physical inventory is zero. Counterexamples to the optimality of this rule have been found but are rare (see Maxwell [17] and Delporte and Thomas [3]).

Two approaches for heuristic algorithms exist, the basic period approach and the time-varying lot sizes approach. The basic period approach requires, in addition to the ZSR, every item to be produced at equally-spaced intervals of time that are multiples of a basic

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time period. (This together with the ZSR implies that every item is produced in equal lot sizes.) Most of the heuristic algorithms that follow this approach first select the *frequency* (i.e., number of production runs per cycle) with which each item is to be produced, and then search for a feasible schedule that implements these frequencies. See Elmaghraby [5] for an excellent review on this approach up to late 1970s. Under this approach it is NP-complete to determine the existence of a feasible schedule (see Hsu [13]). These difficulties have led some researchers to reject the basic-period paradigm, in particular the requirement of equally spaced production lots.

The time-varying lot sizes approach, which relaxes the restriction of equally spaced production runs, was initiated by Maxwell [17] and Delporte and Thomas [3]. Dobson [4] showed that any *production sequence* (i.e., the order in which the items are produced in a cycle) can be converted into a feasible production schedule in which the quantities and timing of production lots are not necessarily equal provided that the proportion of time available for setups is positive. Dobson also developed a heuristic to generate production frequencies and a sensible production sequence. Near optimal schedules can be obtained by combining Dobson's heuristic with Zipkin's [29] algorithm which finds an optimal schedule for a given production sequence. Gallego and Roundy [9] extended the time-varying lot sizes approach to the ELSP which allows backorders. Gallego and Shaw [10] showed that the ELSP is strongly NP-hard under the time-varying lot sizes approach with or without the ZSR restriction, giving theoretical justification to the development of heuristics.

In this article we are primarily concerned with highly constrained facilities where the production capacity constraint is binding. By this we mean facilities where it is *not* optimal to insert idle times between the production of different items. Highly constrained facilities have large lot sizes which are driven by their setup times through the dual variable associated with the capacity constraint. This typically results in very high long run average holding costs relative to the long run average setup costs. By reducing setup *time*, lot sizes are significantly reduced resulting in a better balance between the long run average holding and setup costs.

Most researchers have concentrated on the study of setup *cost* reduction models (See Hall [12], Porteus [20, 21]). However, as Dobson [4], Karmarkar [15], and Sheldon [23] have observed, there is often no real out-of-pocket setup costs in the sense of cash flows being affected. Rather, the setup costs are only a surrogate for the violation of capacity constraints. Moreover, most industrial applications deal with setup time reduction, see Shingo [24]. Our motivation is to model setup times and production capacity explicitly in the hope of gaining insights about the economic impact of setup time reductions. Since we will model setup times and production capacity explicitly, there is no need to include surrogate setup costs. Consequently, if there are no out-of-pocket expenses directly related to the setup operations we model setup costs as zero. In what follows, by setup costs we mean exclusively out-of-pocket expenses incurred as a result of performing setup operations.

Beek and Putten [1] described how OR models can contribute to quantify the integral effects of investment decisions with respect to production systems. They gave several examples illustrating opportunities to reduce setup times, to increase the production capacity, and to cut supply leadtimes, etc. In addition to setup time reductions, Beek and Putten suggested increasing the production rate. However, increasing the production rate is often impossible because of technical constraints, or expensive compared to the cost of setup time reduction. Because of this reason, we only study the effect of investment decisions in reducing setup times.

Spence and Porteus [26] applied the setup time reduction concept to the multi-item capacitated EOQ model of Hadley and Whitin [11], which is a simpler version of the ELSP since it ignores the scheduling issue. Gallego and Moon [8] developed a model which considers the economic effects of externalizing internal setup operations in the ELSP context. The tradeoff in [8] is between the decrease in holding costs resulting from shorter setup times, and the higher out-of-pocket setup costs resulting from performing setup operations off-line. Externalizing internal setup operations result in higher setup costs because such operations are typically more time consuming when done off-line—they require additional or better trained workers, or more careful coordination by management. Perhaps the most important observation in [8] is that lot sizes may decrease as setup costs increase. This observation seems to be at odds with traditional inventory theory, where lot sizes are proportional to the square root of their setup costs. The explanation is that for highly constrained facilities, the lot sizes are mainly driven by the setup times, and reducing them can result in smaller lot sizes even when setup costs are increased. This explains why some Japanese companies have been willing to spend more on setup costs to reduce internal setup times in order to reduce lot sizes and average cost. In effect, they are *trading setup time for setup cost.*

In this article, we assume that setup times can be reduced by a one time investment. One time investments to reduce setup time operations include investment on special tools and equipment, as well as in software for numerically controlled machines. The key decision is to determine the magnitude of the one time investment, and the allocation of this investment among the different setup operations. Thus, the main difference between our model and that in [8] is in the way we model the cost of reducing setups. In [8], internal setup operations are externalized, and the cost of externalizing is reflected in higher out-of-pocket setup costs. In our model, there is a one time cost of reducing setup times. Although the formulations of these two problems are similar, the analysis required to study them is quite distinct.

Note that under the average cost criterion, it is justified to make a one time investment of $1,000,000 in reducing setup times if this investment leads to a reduction of the annual cost by a positive amount, say one dollar. This is because any finite investment becomes negligible under the average cost criterion.

Of course, we would not recommend a $1,000,000 investment in reducing average cost unless the average cost is reduced by more than the opportunity cost of the $1,000,000. Thus, it seems reasonable to select the level of the one time investment to minimize the sum of the annual average cost plus the annual opportunity cost of the investment. Yet in doing so we are using an undiscounted average cost criterion for the inventory carrying and setup cost while amortizing the one time investment. This may be dismissed as adding apples and oranges, but we will argue that this practice is approximately consistent with the annual worth criterion.

To see this, consider a one time investment of c dollars that results in an annual cost of $a(c)$. If the discount rate is α, then the present value of the one time investment plus an infinite sequence of $a(c)$ yearly costs is given by $a(c)/\alpha + c$, and minimizing this expression with respect to c is equivalent to minimizing $a(c) + \alpha c$ which is the cost per unit time plus the amortized cost αc of the one time investment. If, however, $a(c)$ is not the exact cost rate, but only the long run average cost of a cash flow, say $a(c, t)$, then adding the amortized cost αc of the one time investment is not equivalent to the annual cost criterion. This may be a good approximation, however, when $a(c, t)$ is periodic with a sufficiently small period, say T, and when $a(c, t)\, 0 \leq t \leq T$ does not vary much from $a(c)$. In this case,

$$a(c) + \alpha c,$$

is a surrogate for the more complicated expression

$$\frac{\int_0^T a(c, t) e^{-\alpha t} dt}{1 - e^{-\alpha T}} + c$$

for annual worth. By the Cauchy-Schwartz inequality $a(c) + \alpha c$ is an upper bound on this expression.

In the ELSP, the synchronization constraint forces the production of the items to be scattered in time over a cycle, so the actual cost rate does not deviate much from the long run average cost. Thus, adding the amortized cost of the one time investment to the long run average cost is a good approximation to the equivalent annual worth criterion. While this approximation is not exact, the alternative is to deal with a much more complicated cost function from which it would be difficult to derive meaningful insights.

When setup costs are zero, or are proportional to the setup times, we show that a lower bound of the long run average cost is *increasing concave* in the setup times. This lower bound on the long run average cost, will be called hereafter *the lower bound cost.* For general production sequences, it is possible to obtain feasible schedules by using Dobson's algorithm [4], whose cost is on the average about 4% higher than the lower bound cost.

The increasing concave property means that the lower bound cost drops more steeply the more we reduce the setup times. That is, the reduction in the lower bound cost resulting from the reduction of the setup times gets steeper (in the weak sense) the more we reduce the setup times. This helps explain why reducing setup times have been viewed by some as the best investment in manufacturing.

We also show that when the capacity constraint is binding, the reorder intervals from the lower bound (which may be infeasible due to the relaxation of the synchronization constraint) and the lower bound cost are homogeneous functions of degree one in the setup times. This means that doubling (resp., halving) the setup times results in doubling (resp., halving) the reorder intervals and the lower bound cost, i.e., both functions act linearly on straight lines emanating from the origin. This result is important because it dispels the common belief that the reorder intervals and the average cost grow with the square root of the setup times. This belief is rooted in the practice of using surrogate setup costs that are proportional to the setup times. This practice is incorrect, because it ignores the fact that the dual variable associated with the capacity constraint, which relates surrogate setup costs to setup times, decreases as the setup times decrease.

A heuristic that generates cyclic schedules is obtained by first finding the setup times that minimize the lower bound on the average inventory carrying and setup cost plus the amortized cost of reducing the setup times. Then, for these setup times, we obtain reorder intervals ignoring the synchronization constraints and round them to powers-of-two multiples of a base planning period as suggested by Roundy [22]. This gives rise to relative production frequencies which can then be used to obtain a production sequence using Dobson's [4] heuristic. Optimal time-varying lot sizes corresponding to these frequencies can be computed by using Zipkin's [29] algorithm. Numerical examples indicate that significant savings are possible for highly utilized facilities.

This article is organized as follows: In Section 2 we introduce notation and state our assumptions. In Section 3 we develop a lower bound on the average cost as a function of the setup times. When the setup costs are zero or proportional to the setup times, we obtain a *closed* form expression for the lower bound and show that it is increasing concave in the setup times. We also show that the lower bound cost and the reorder intervals that achieve it are homogeneous of degree one in the setup times. In addition, we show that if it is equally costly to reduce the setup times, then it is most profitable to reduce the setup time of the item that is produced most frequently. In Section 4, we consider the problem of minimizing the lower bound cost plus the amortized cost of the one time investment. We show that a global minimum for this problem always exists, and that under certain conditions there is a unique global minimum. In this section we also discuss the special case of common cycle schedules. Numerical results are reported in Section 5, and our conclusions in Section 6.

2. NOTATION AND ASSUMPTIONS

The data for the problem are:

Index for the items		$i = 1, \ldots, m,$
Lower bound on setup times (day)	$\underline{s}_i$	$i = 1, \ldots, m,$
Initial setup times (day)	$\bar{s}_i$	$i = 1, \ldots, m,$
Initial setup costs ($)	$\bar{A}_i$	$i = 1, \ldots, m,$
Constant production rates (unit/day)	p'_i	$i = 1, \ldots, m,$
Known inventory holding costs ($/unit, day)	h'_i	$i = 1, \ldots, m,$
Constant demand rates (unit/day)	d'_i	$i = 1, \ldots, m.$

By initial setup times and setup costs, we mean the setup times and setup costs before the investment in reducing setup times. Without loss of generality we redefine the units of the items so demand rates are all equal to one. This is accomplished by setting $d_i \equiv d'_i/d'_i = 1, p_i \equiv p'_i/d'_i$ and $h_i \equiv h'_i d'_i$. The transformation maps many equivalent problems to one that is easier to manipulate. For convenience, define $H_i \equiv 1/2h_i(1 - 1/p_i)$. Let T_i denote order interval of item i, $i = 1, \ldots, m$. We denote vectors by bold faced letters; for instance $\mathbf{s} = (s_1, \ldots, s_m)$, etc.

Define $k \equiv 1 - \sum_{i=1}^{m} (1/p_i)$. Note that k is the long run proportion of time available for setups. For infinite horizon problems $k > 0$ is a necessary and sufficient condition for the existence of a feasible schedule, see Dobson [4].

As discussed earlier, there are often no out-of-pocket setup costs, and in this case we model the setup costs as zero. More generally, however, there may be positive out-of-pocket setup costs, which we model as fixed plus linear in the setup times. Thus, the setup cost of item i is modeled as $A_i(s_i) = K_i + \beta_i s_i$, where $K_i \geq 0$ is a fixed out-of-pocket cost for the setup of item i and $\beta_i \geq 0$ is the out-of-pocket cost charged per unit of time spent in the setup of item i. The fixed part K_i may represent a fixed out-of-pocket cost that must be paid every time a setup of product i is performed, i.e., the cost of materials consumed during the setup (so long as the quantity is independent of the setup time), while β_i may be the direct labor cost per unit time associated with the setup of item i. We assume that β_i includes all costs that vary with time, for instance labor cost, except the cost of machine down time.

Clearly $K_i = \bar{A}_i - \beta_i \bar{s}_i$ where $\bar{A}_i$ and $\bar{s}_i$ are respectively the *initial* setup cost and the *initial* setup time. Notice that our formulation allows lower bounds on the setup times which may exist for technological reasons; if no such bounds exist then the $\underline{s}_i$'s can be set to zero.

We now turn our attention to the one time cost of reducing the setup times. Let $c : \times_{i=1}^{m} [\underline{s}_i, \bar{s}_i] \to \mathcal{R}_+$ denote the one time cost of reducing the setup times to $\mathbf{s}$. To facilitate the analysis, we assume that $c(\cdot)$ is twice continuously differentiable and strictly decreasing. We assume $c(\cdot)$ to be convex in any direction of reduction of setup times. This assumption is reasonable since we expect further decrease in any particular direction to become more and more expensive. This implies that $c(\cdot)$ itself is convex when $c(\cdot)$ is separable, i.e., of the form $c(\mathbf{s}) = \sum_{i=1}^{m} c_i(s_i)$. Separability of the cost function is appropriate when setup reductions are product dependent, such as when the setup for a product consists of installing a unique guide tool. (See Spence and Porteus [26].) We denote $\partial c(s_1, \ldots, s_m)/\partial s_i$ by $c_i(s_1, \ldots, s_m)$, and $\partial^2 c(s_1, \ldots, s_m)/\partial s_i \partial s_j$ by $c_{i,j}(s_1, \ldots, s_m)$, respectively. However, when $c(\mathbf{s})$ is separable, we denote the partial derivative of $c(\mathbf{s})$ with respect to s_i by $c_i'(s_i)$, and this should cause no confusion.

In general, assuming $c(\cdot)$ to be convex implies that if two different vectors of setup times are achievable at a certain cost, then all the vectors that lie in the straight line that joins them must also be achievable at the same cost. Since this may not always be the case, one should use caution in using the convexity assumption.

A comment on the continuity assumption on $c(\cdot)$ is needed. In reality the set of investments that reduce setup times may be a finite set. However, there are many setup procedures which have 20 or more substages where each stage can be reduced independent of the other stages. See for example the setup procedure for the cold-forging machine, (Shingo [24]). Let n be the total number of substages for a setup procedure. Then the number of possible set of investments is $\sum_{k=0}^{n} nC_k = 2^n$, a large number which allows a continuous approximation of the resulting function.

Commonly used cost functions in the economic literature are of the *Cobb-Douglas* type $c(s_1, \ldots, s_m) = A \prod_{i=1}^{m} (\bar{s}_i - s_i)^{a_i}$ where $A > 0$, $a_i > 1$ for all i. (See Varian [28] for details.) Spence and Porteus [26] argued that separable cost functions $c(\mathbf{s}) = \sum_i c_i(s_i)$ where $c_i(s_i) = a_i s_i^{-b_i} - d_i$ arise in practice when setup reduction is item dependent. A detailed analysis of the separable case is deferred to Section 4 and computational results to Section 5.

The problem can be now stated as follows. There is a single facility on which m distinct items are to be produced. We try to find a cycle length T, a production sequence $f^1, \ldots, f^n (f^j \in \{1, \ldots, m\})$, $n \geq m$, setup times $s_1, \ldots, s_m$ (consequently setup costs $a_1, \ldots, a_m$), production times $t^1, \ldots, t^n$ and idle times $u^1, \ldots, u^n$ so that the production sequence is executable in the chosen cycle length, the cycle can be repeated indefinitely, demand is met and total (amortized investment plus inventory and setup) cost per unit time is minimized.

3. A LOWER BOUND ON THE AVERAGE HOLDING AND SETUP COST

In this section we develop a lower bound on the long run average holding and setup cost. In the next section, we will consider the problem of minimizing this lower bound plus the amortized cost of the one time investment in reducing setup times.

For fixed setup times $\mathbf{s}$, the lower bound problem is given below:

$$LB(\mathbf{s}) = \min_{T_1, \ldots, T_m} \sum_{i=1}^{m} \left[\frac{A_i(s_i)}{T_i} + H_i T_i \right]$$

$$\text{subject to} \quad \sum_{i=1}^{m} \frac{s_i}{T_i} \le k$$
$$T_i \ge 0 \qquad i = 1, \ldots, m.$$

The objective function denotes the average holding and setup cost per unit time. The capacity constraint is explicitly considered. However the synchronization constraint, stating that no two items can be scheduled to produce at the same time, is ignored. Consequently, the value of the program results in a lower bound on the average cost over all cyclic schedules, see Bomberger [2].

Let λ denote the dual variable for the capacity constraint. By optimizing over $(T_1, \ldots, T_m)$ for fixed λ, we obtain the Stöer dual, see Stöer [27],

$$LB(\mathbf{s}) = \max_{\lambda \ge 0} LB(\mathbf{s}, \lambda)$$

where

$$LB(\mathbf{s}, \lambda) = 2 \sum_{i=1}^{m} \sqrt{H_i(A_i(s_i) + \lambda s_i)} - k\lambda.$$

We have long suspected $LB(\mathbf{s})$ to be increasing concave in $\mathbf{s}$. Notice that a sufficient condition for $LB(\mathbf{s})$ to be concave is that $LB(\mathbf{s}, \lambda)$ be concave. Unfortunately $LB(\mathbf{s}, \lambda)$ is not concave even when the K_i's and the β_i's are all zero (the hypo-graph is not convex). On the other hand the concavity of $LB(\mathbf{s}, \lambda)$ in λ for fixed $\mathbf{s}$ is not sufficient. Nor is the fact that $LB(\mathbf{s}, \lambda)$ is concave in $\mathbf{s}$ for fixed λ. Under these conditions it is very difficult to prove the concavity of $LB(\mathbf{s})$. Nevertheless, we are able to show that $LB(\mathbf{s})$ is indeed concave when the out-of-pocket setup costs are proportional to the setup times, i.e., when $A_i(s_i) = \beta s_i$. This type of setup cost arises when a team of operators is paid a fixed hourly rate for setups regardless of the set of operations they perform. This model encompasses the case of zero out-of-pocket setup costs discussed by Dobson [4], Karmarkar [15], and Sheldon [23]. The concavity of $LB(\mathbf{s})$ under more general conditions remains an open question.

LEMMA 1. If $A_i(s_i) = \beta s_i$ for all $i = 1, \ldots, m$ and the capacity constraint is binding, then the optimal reorder intervals for the lower bound, and the non-constant portion of $LB(\mathbf{s})$ are homogeneous functions of degree one.

Before presenting the proof, notice that when $A_i(s_i) = \beta s_i$ for all $i = 1, \ldots, m$, $LB(\mathbf{s}, \lambda)$ reduces to

$$LB(\mathbf{s}, \lambda) = 2 \sum_{i=1}^{m} \sqrt{(\beta + \lambda) H_i s_i} - k\lambda.$$

Recall that the Stöer dual calls for maximizing $LB(\mathbf{s}, \lambda)$ over $\lambda \ge 0$, and since $LB(\mathbf{s}, \lambda)$ is concave in λ, it follows that λ^*, the maximizer of $LB(\mathbf{s}, \lambda)$, is positive if and only if the partial derivative of $LB(\mathbf{s}, \lambda)$ with respect to λ is positive at $\lambda = 0$. This is so, if and only if

$$\sum_{i=1}^{m} \sqrt{\frac{H_i s_i}{\beta}} - k > 0.$$

By complementary slackness, the capacity constraint is binding whenever $\lambda^* > 0$. Solving for β we see that the capacity constraint is binding whenever

$$\beta < \left(\frac{\sum_{i=1}^{m} \sqrt{H_i s_i}}{k} \right)^2.$$

In particular, the capacity constraint is binding when $\beta = 0$.

PROOF. If the capacity constraint is binding it is possible to solve the Stöer dual in closed form. To see this, notice that if the capacity constraint is binding then there exists a $\lambda^* \geq 0$ such that

$$\sum_{i=1}^{m} \sqrt{\frac{H_i s_i}{\beta + \lambda^*}} - k = 0.$$

Consequently,

$$\beta + \lambda^* = \left(\frac{\sum_{i=1}^{m} \sqrt{H_i s_i}}{k} \right)^2,$$

resulting in

$$T_i(\mathbf{s}) = \sqrt{\frac{(\beta + \lambda^*) s_i}{H_i}} = \frac{1}{k} \left(\sum_{j=1}^{m} \sqrt{H_j s_j} \right) \sqrt{\frac{s_i}{H_i}},$$

and

$$LB(\mathbf{s}) = \frac{1}{k} \left(\sum_{j=1}^{m} \sqrt{H_j s_j} \right)^2 + \beta k.$$

It is clear that $T_i(\mathbf{s})$ and the nonconstant part of $LB(\mathbf{s})$, namely

$$g(\mathbf{s}) = \frac{1}{k} \left(\sum_{j=1}^{m} \sqrt{H_j s_j} \right)^2$$

are homogeneous of degree one, i.e., of the form $f(\gamma \mathbf{x}) = \gamma f(\mathbf{x})$, $\gamma \geq 0$. ■

We are now ready to state:

PROPOSITION 1: If the setup costs are proportional to the setup times, i.e., if $A_i(s_i) = \beta s_i$ holds for all $i = 1, \ldots, m$, then the lower bound cost $LB(\mathbf{s})$ is increasing concave in $\mathbf{s}$.

PROOF: There are two cases to consider. If the capacity constraint is not binding, then $\lambda^* = 0$, and $LB(\mathbf{s}) = LB(\mathbf{s}, 0) = 2 \sum_{i=1}^{m} \sqrt{\beta H_i s_i}$ is clearly increasing concave in $\mathbf{s}$. On the other hand, if the capacity constraint is binding, then $LB(\mathbf{s}) = g(\mathbf{s}) + \beta k$. Clearly $LB(\mathbf{s})$ is increasing in $\mathbf{s}$. To prove concavity of $LB(\cdot)$ it is enough to show that $g(\cdot)$ is concave. To see this, we first show that $g(\mathbf{x}+\mathbf{y}) \geq g(\mathbf{x}) + g(\mathbf{y})$. This is true since

$$g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) - g(\mathbf{y}) = \frac{1}{k} \sum_{i<j} \sqrt{H_i H_j} \sqrt{(\sqrt{x_i y_j} - \sqrt{x_j y_i})^2} \geq 0$$

Since $g(\mathbf{x})$ is homogeneous of degree 1, we have

$$g(\gamma\mathbf{x} + (1-\gamma)\mathbf{y}) \geq \gamma g(\mathbf{x}) + (1-\gamma) g(\mathbf{y})$$

establishing the concavity of $g(\cdot)$. ■

Since Dobson's heuristic results in feasible schedules with average costs that are very close to $LB(\mathbf{s})$, our result suggests that the average cost is likely to drop more steeply the more we reduce the setup times, i.e., reducing setup times may have increasing marginal returns. Informally this means that for any direction $d = (d_i)_{i=1}^m$, $d_i > 0$, the cost savings obtained from reducing $\bar{s}$ to $\bar{s} - d$ are not larger than the cost savings obtained from reducing $\bar{s} - d$ to $\bar{s} - 2d$. Notice that if d is in the direction of $\bar{s}$, so that $\bar{s} - d$ is proportional to $\bar{s}$, then the cost savings obtained from reducing $\bar{s}$ to $\bar{s} - d$ are *equal* to the cost savings obtained from reducing $\bar{s} - d$ to $\bar{s} - 2d$.

We will now show that if it is equally costly to reduce each of the setup times, then larger costs savings accrue from reducing the setup time of the item that is produced most frequently.

COROLLARY 1: When the setup costs are proportional to the setup times, reducing the setup time of the item with the smallest reorder interval has the largest impact in reducing the lower bound on the long run average cost.

PROOF: It is easy to see that the derivative of $LB(\mathbf{s})$ with respect to s_i is proportional to the reciprocal of the reorder interval T_i, this being true regardless of whether the capacity constraint is binding or not. Hence the derivative is steepest for the item with the smallest reorder interval, i.e., the item produced most frequently. ■

The intuition is that reducing the setup time of the item that is produced most frequently (smallest T_i) has the highest return since we enjoy the benefits of the setup time reduction every time that item is produced. It is interesting to observe that if the holding costs are equal, then the most frequently produced item is the one with the smallest setup time. This is in sharp contrast to the common practice of reducing the *longest* setup time.

4. MINIMIZING THE LOWER BOUND PLUS THE AMORTIZED COST

Incorporating the one time investment cost $c(\mathbf{s})$ our objective is to minimize

$$LB(\mathbf{s}) + \alpha c(\mathbf{s})$$

subject to

$$\underline{s} \leq s \leq \bar{s}.$$

The existence of a global minimizer is guaranteed because we are minimizing a continuous function

$$f(\mathbf{s}) \equiv LB(\mathbf{s}) + \alpha c(\mathbf{s})$$

over a compact set

$$D \equiv \{\mathbf{s} : \underline{s} \leq s \leq \bar{s}\}.$$

There are many commercial codes available to solve nonlinear programs over compact sets. Most of the codes stop upon finding a local minimizer. For this reason it is important to know when there is a unique local, and hence global, minimizer.

We know two ways of showing when a function has a unique local, and hence, global minimizer. One is for the function to be connected. The other is by showing that there is a unique Karush-Kuhn-Tucker (KKT) point.

Recall that a point is a local minimizer if there exists an open neighborhood around the point over which no other point has a lower value function. A local minimizer is proper if the value function at the point is strictly lower than the value function of all other points in the neighborhood. It is not well known that a sufficient condition to guarantee the uniqueness of a local (resp., proper) minimizer is that the objective function be connected (resp., strictly connected) over the feasible region, see Ortega and Rheinboldt [19].

A function $g(\mathbf{x})$ is said to be connected over the set C if for every $\mathbf{x}$ and $\mathbf{y}$ in C, there exists a continuous function $p : [0, 1] \to C$ such that $p(0) = \mathbf{x}$, $p(1) = \mathbf{y}$ and $g(p(t)) \leq \max\{g(\mathbf{x}), g(\mathbf{y})\}$. A connected function is said to be strictly connected if the inequality holds strictly for $0 < t < 1$. When $p(t)$ is the straight line between $\mathbf{x}$ and $\mathbf{y}$, e.g., $p(t) = (1 - t)\mathbf{x} + t\mathbf{y}$, the function $g(\mathbf{x})$ is said to be quasi-convex. Thus all quasi-convex functions are connected.

$LB(\mathbf{s})$ being increasing concave is connected, since for any two points $\mathbf{s}$ and $\mathbf{s}'$ the path that goes straight from $\mathbf{s}$ to the origin, and from the origin to $\mathbf{s}'$ is a connected path over which $LB(\cdot)$ is at most the maximum of $LB(\mathbf{s})$ and $LB(\mathbf{s}')$. If $c(\mathbf{s})$ is convex, then it is quasi-convex. If $c(\mathbf{s})$ is concave, then it is connected on account of being decreasing. Thus, $c(\mathbf{s})$ is connected regardless of whether $c(\mathbf{s})$ is convex or concave. While we cannot claim that the sum of any two connected functions is connected, the result is true under fairly mild conditions.

PROPOSITION 2: Let $f(\cdot)$ and $g(\cdot)$ be any two connected functions. If for any $\mathbf{x}$, $\mathbf{y}$ the level set $\{\mathbf{z} : f(\mathbf{z}) = f(\mathbf{x})\}$ and the level set $\{\mathbf{z} : f(\mathbf{z}) = f(\mathbf{y})\}$ each intersects the level sets $\{\mathbf{z} : g(\mathbf{z}) = g(\mathbf{x})\}$ and $\{\mathbf{z} : g(\mathbf{z}) = g(\mathbf{y})\}$ then $f(\cdot) + g(\cdot)$ is connected.

PROOF: See Appendix.

If the condition of Proposition 2 is met for $LB(\mathbf{s}) + \alpha c(\mathbf{s})$, then there is a unique global minimum and any commercial nonlinear programming code can be used to minimize the lower bound cost plus the amortized cost of the one-time investment.

If the above condition fails, or if the above condition is difficult to verify then there is an alternative well known method of establishing that there is a unique local, and hence global, minimizer. The method consist of examining the Karush-Kuhn-Tucker (KKT) points. Recall that a KKT point is either an interior point at which the gradient of the objective function vanishes, or it is a boundary point at which certain complementary conditions hold. Recall also that all local minimizers, if regular, must satisfy the Karush-Kuhn-Tucker conditions, see Luenberger [16]. There are different constraint qualifications that qualify a point as regular. One sufficient condition for a point to be regular is that the gradients of the active constraints are linearly independent. The constraints in our problem are lower and upper bounds on the s_i's. The gradient of these constraints are the unit vectors in either the positive or the negative direction. Since the lower and the upper bound constraint can not be active at the same time, the gradients of the active constraints must be linearly independent. Thus, the constraint qualification holds for our problem. Now, a Karush-Kuhn-Tucker point is a local minimizer if it satisfies certain second order conditions. Once it is known that a minimizer exists, and that the constraint qualification holds at each minimizer, then the existence of a *unique* KKT point implies the existence of a *unique* local, and hence global, minimizer. Notice that when there is a unique KKT point there is no need to verify the second order conditions, since they must be satisfied a-fortiori.

We will use the above to show that if the cost of reducing the setup times is separable, then there is a unique KKT point. To do so, we first write the KKT conditions for the general problem

$$\min_{\mathbf{s}\in D} [LB(\mathbf{s}) + \alpha c(\mathbf{s})].$$

Recall that $LB(\mathbf{s})$ is the solution of the Stoër dual

$$LB(\mathbf{s}) = \max_{\lambda\geq 0} LB(\mathbf{s}, \lambda).$$

Relaxing the constraint on the setup times results in the Lagrangean function

$$L(\mathbf{s}, \lambda, \mu, v) = LB(\mathbf{s}, \lambda) + \alpha c(\mathbf{s}) + \mu'(\underline{s} - s) + v'(s - \bar{s}),$$

where $v = (v_i)_{i=1}^m$, and v_i is the dual variable corresponding to $s_i \leq \bar{s}_i$, and $\mu = (\mu)_{i=1}^m$, where μ_i is the dual variable corresponding to $\underline{s}_i \leq s_i$.

The Karush-Kuhn-Tucker conditions are then given by:

$$LB_i(\mathbf{s}, \lambda) + \alpha c_i(\mathbf{s}) + \mu_i - v_i = 0, \tag{1}$$

$$\lambda \geq 0 \qquad \text{complementary slackness (c.s.) with } LB_\lambda(\mathbf{s}, \lambda) \leq 0 \tag{2}$$

$$v_i \geq 0 \qquad \text{c.s. with} \qquad s_i \leq \bar{s}_i \tag{3}$$

and

$$\mu_i \geq 0 \qquad \text{c.s. with} \qquad \bar{s}_i \leq s_i \tag{4}$$

where

$$LB_i(\mathbf{s}, \lambda) = (\lambda + \beta_i)\sqrt{\frac{H_i}{K_i + (\beta_i + \lambda)s_i}}$$

is the partial derivative of $LB(\mathbf{s}, \lambda)$ with respect to s_i and

$$LB_\lambda(\mathbf{s}, \lambda) = \sum_{i=1}^{m} s_i \sqrt{\frac{H_i}{K_i + (\beta_i + \lambda)s_i}}$$

is the partial derivative of $LB(\mathbf{s}, \lambda)$ with respect to λ.

For the special case where $c(\mathbf{s}) = \sum_{i=1}^{m} c_i(s_i)$, we replace $c_i(\mathbf{s})$ by $c_i'(s_i)$. Let $g_i(\cdot)$ denote the inverse function of $c_i'(\cdot)$. We are now ready to show

PROPOSITION 3: If $c(\mathbf{s})$ is separable and strictly convex, and for fixed λ the function

$$s_i = \Gamma_i(s_i, \lambda) \equiv \min\left\{\max\left[\underline{s}_i, g_i\left(-\frac{LB_i(\mathbf{s}, \lambda)}{\alpha}\right)\right], \bar{s}_i\right\}$$

admits a unique fixed point for each $i = 1, \ldots, m$, then there is a unique KKT point, and such a point is a global optimum.

PROOF: Let $\mathbf{s} = (s_i)_{i=1}^{m}$ be the vector of fixed points of $\Gamma_i(s_i, \lambda)$, $i = 1, \ldots, m$. Then there exists nonnegative vectors μ and ν, depending on λ such that conditions (1), (3), and (4) are satisfied. It is easy to see that $LB_i(\mathbf{s}, \lambda)$ is increasing in λ which renders $\Gamma_i(s_i, \lambda)$ decreasing in λ. Thus, the fixed points are decreasing in λ. Consequently, to satisfy constraint (2), all that is needed is a line search in λ starting from $\lambda = 0$. The convergence of the line search is guaranteed by the continuity of the $\Gamma_i(s_i, \lambda)$ and of $LB_\lambda(\mathbf{s}, \lambda)$. Since there is a unique regular KKT point, this must be the global minimizer of $f(\mathbf{s})$ over D. ■

REMARK 1: It is easy to see that for fixed λ, the function $LB_i(\mathbf{s}, \lambda)$ is decreasing in s_i, while $g_i(\cdot)$ is strictly increasing. Thus, the function $\Gamma_i(\cdot, \lambda)$ is strictly increasing over $[\underline{s}_i, \bar{s}_i]$. This implies that $\Gamma_i(s_i, \lambda)$ must have at least one fixed point. The requirement of the proposition is met for example when $\Gamma_i(\underline{s}_i, \lambda) > \underline{s}_i$ and $\Gamma_i(\cdot, \lambda)$ has at most one fixed point on $s_i \leq \bar{s}_i$. The requirement is also met if $\Gamma_i(\underline{s}_i, \lambda) < \underline{s}_i$ and $\Gamma_i(\cdot, \lambda)$ does not have any fixed point on $\underline{s}_i \leq s \leq \bar{s}_i$.

REMARK 2: It is easy to check directly that the unique KKT point is indeed a strict local, and hence strict global minimizer, by verifying that the Hessian of the Lagrangean function is positive definite on the tangent plane of the active constraints.

When the conditions of Proposition 3 are satisfied, we can solve the problem by the following algorithm. To facilitate the presentation of the algorithm, we denote by $\mathbf{s}_\lambda$ the vector of fixed points of $\Gamma_i(s_i, \lambda)$, $i = 1, \ldots m$.

Algorithm

Step 1. (Check if $\lambda = 0$ gives an optimal solution.)
Stop if $LB_\lambda(\mathbf{s}_0, 0) \leq 0$. Else go to *Step 2.*

Step 2. (Trapping the optimal λ.)
Set $\lambda_l = 0$ and $\lambda_h = 1$.
Evaluate $LB_\lambda(\mathbf{s}_{\lambda_h}, \lambda_h)$ doubling λ_h if necessary until $LB_\lambda(\mathbf{s}_{\lambda_h}, \lambda_h) < 0$.
If $\lambda_h > 1$, set $\lambda_l = 0.5\,\lambda_h$. The optimal value of λ lies in the interval $[\lambda_l, \lambda_h]$.

Step 3. (Line search on λ)
Let $\epsilon > 0$, and $d = 1$. Do until $d < \epsilon$. Let $\lambda_m = 0.5(\lambda_l + \lambda_h)$.
If $LB_\lambda(\mathbf{s}_{\lambda_m}, \lambda_m) < 0$ set $\lambda_h = \lambda_m$.
Otherwise set $\lambda_d = \lambda_m$. Let $d = \lambda_h - \lambda_l$.

At the end of *Step 3* we have a KKT point that minimizes the lower bound cost plus the amortized cost of the one-time investment. The optimal setup times are given by $\mathbf{s} = \mathbf{s}_{\lambda_m}$, the optimal dual variable by $\lambda^* = \lambda_h$, and the optimal cycle lengths by

$$T_i = \sqrt{\frac{K_i + \beta_i s_i + \lambda s_i}{H_i}}\,.$$

An Optimal Common Cycle Schedule

A common cycle (CC) schedule is a cyclic schedule where all the items are produced exactly once per cycle. Common cycle schedules perform adequately in some realistic situations. (See Jones and Inman [14], and Gallego [6].) A lower bound on the cost of CC schedules is obtained by solving the following program:

$$LBCC(\mathbf{s}) = \min_T \sum_{i=1}^{m} \left[\frac{A_i(s_i)}{T} + H_i T\right]$$

$$\text{subject to} \quad \sum_{i=1}^{m} \frac{s_i}{T} \leq k$$

$$T \geq 0 \qquad i = 1, \ldots, m.$$

The cost $LBCC(\mathbf{s})$ is a lower bound on the cost of CC schedules for the same reason $LB(\mathbf{s})$ is a lower bound on the cost of all cyclic schedules. It turns out, however, that it is always possible to satisfy the synchronization constraint when $T_i = T$ for all $i = 1, \ldots, m$. Consequently, it is possible to find a CC schedule with cost $LBCC(\mathbf{s})$.

Incorporating the one time investment cost $c(\mathbf{s})$ our objective is to minimize

$$LBCC(\mathbf{s}) + \alpha c(\mathbf{s})$$

subject to

$$\underline{s} \le s \le \bar{s}.$$

If the conditions of Proposition 2 are met, then any commercial code can be used to solve for a common cycle schedule. Otherwise, the algorithm of Section 4 can be adapted to obtain the optimal common cycle schedule provided the conditions of Proposition 3 are met.

Here we derive succinct conditions for the case where the K_i's are all zero, and $c(\mathbf{s}) = \sum_{i=1}^{m} c_i(s_i)$ is separable and convex. If β is zero, or sufficiently small so that the capacity constraint is binding, then the long run average holding and setup cost is *linear* in $\mathbf{s}$, i.e.,

$$LB(\mathbf{s}) = H \sum_{i=1}^{m} s_i + k\beta$$

where $H \equiv \sum_{i=1}^{m} H_i/k$, and the average cost plus the amortized cost of the one-time investment is separable and convex

$$f(\mathbf{s}) = \sum_{i=1}^{m} (Hs_i + \alpha c_i(s_i)).$$

It follows that minimizing $f(\mathbf{s})$ over D is equivalent to minimizing m one dimensional convex functions over a closed interval. Letting $g_i(\cdot)$ denote the inverse function of $c_i'(\cdot)$, the optimal solution is given by

$$s_i = \min\left\{\max\left[\underline{s}_i, g_i\left(-\frac{H}{\alpha}\right)\right], \bar{s}_i\right\},$$

for $i = 1, \ldots, m$. For example, if $c_i(s_i) = 1/s_i$, then

$$s_i = \min\left\{\max\left[\underline{s}_i, \sqrt{\frac{\alpha}{H}}\right], \bar{s}_i\right\}.$$

5. NUMERICAL EXAMPLES

The 10-item problem in Bomberger [2] has been used extensively to study the performance of heuristics for the ELSP. Very often researchers scale up the basic demands in Bomberger's example to achieve different load factors. Here we consider Bomberger's example with the basic demand scaled by a factor of four, which results in $k = 0.12$. We also solve the problem with 4.5 times the basic demand resulting in $k = 0.007$. We assume $c(\mathbf{s}) = \sum_{i=1}^{m} c_i(s_i)$ where $c_i(s_i) = a_i s_i^{-b_i} - d_i$. Let θ_i be the cost of reducing the original setup time of item i by 10% and let γ_i be the corresponding compounding parameter. That is, every additional 10% reduction costs $100\gamma_i$% more than the last one. (See Porteus [20] for details.) Then

Table 1. Data for the example.

Product number	$\bar{A}_i$ (\$)	$\bar{s}_i$ (day)	Production rate (unit/day)	Holding cost (\$/unit, day)
1	15	0.125	66.6667	0.0012
2	20	0.125	17.7778	0.0220
3	30	0.250	10.5556	0.0478
4	10	0.125	4.1667	0.0750
5	110	0.500	22.2222	0.1044
6	50	0.250	66.6667	0.0100
7	310	1.000	88.8889	0.0169
8	130	0.500	3.3987	0.9403
9	200	0.750	5.2288	0.1434
10	5	0.125	33.3333	0.0075

$$b_i = \frac{-ln(1+\gamma_i)}{ln(0.9)}, \qquad a_i = \frac{\theta_i \bar{s}_i^{-b_i}}{(0.9)^{-b_i} - 1}, \qquad d_i = a_i \bar{s}_i^{-b_i}.$$

This $c_i(s_i)$ is a convex C^2 function which is zero at $\bar{s}_i$. Let $\theta_i = 500$, $\gamma_i = 0.05$ for all i and $\alpha = 0.001$. For example, it costs \$500 to reduce s_7 from 1 day to 0.9 day. Let $\bar{s}_i$ be the current setup time for all i and $\underline{s}_i$ be $0.4\bar{s}_i$ for all i. Let $\bar{A}_i$ be the initial setup cost for item i. The time unit is day. The holding cost rate is dollars per unit per day. The normalized data for $k = 0.007$ is in Table 1.

At four times the basic demand rates, i.e., $k = 0.12$, it is not profitable to invest in reducing setup times which implies that the capacity constraint is not binding. The results for $k = 0.007$ are summarized in Table 2. For comparison purpose, we include an optimal Common Cycle solution and a near-optimal time-varying lot size ELSP heuristic solution without setup time reductions under the headings CC and ELSP, respectively. We provide the lower bound costs and the actual costs resulting from the heuristic to see the performance of the heuristic. The results reported are for the case $\beta = 0$. Results for $\beta = 30$ were essentially similar, since for such low value of k, the solution is driven by the setup times, not by the setup costs.

We also tested the heuristics for 50 randomly generated problems. The data sets were generated from uniform distribution on the given intervals, i.e., $\bar{s}_i \sim U(0.1, 1)$, $\bar{A}_i \sim U(5, 500)$, $p_i \sim U(4, 40)$, and $h_i \sim U(0.01, 1)$. Since the ELSP is harder when k is small [4] and it is meaningful to invest in reducing setups when k is small, we generated problems by adding products to the problem until k became less than 0.01. On the average, investing in setup times resulted in savings of 32.8% for CC and 23.6% for ELSP compared to the solutions obtained without setup time reduction.

The dramatic savings for $k = 0.007$ and the computational results indicate that setup

Table 2. Computational results for Bomberger's problem ($k = 0.007$).

(\$/day)	CC	CC (reduction)	ELSP	ELSP (reduction)
Setup reduction cost	0	40.04	0	37.18
Holding cost	266.41	115.96	173.73	74.27
Setup cost	1.71	3.93	1.69	3.99
Total average cost	268.12	159.93	175.42	115.44

time reductions are very worth while for highly utilized facilities. Interestingly, the opposite was observed in Silver [25], Gallego [7], and Moon, Gallego, and Simchi-Levi [18] where production rate reduction was shown to be more profitable in under-utilized facilities.

6. CONCLUSIONS

We model a realistic strategic problem of reducing setup times and costs by a one-time initial investment. The objective is to minimize the long run average holding and setup costs plus the amortized investment per unit time. This type of investment has the effect of reducing the lot sizes, in accord with JIT philosophy, and is justified for facilities operating near or at capacity. This approach should be considered as an alternative to an investment in increased capacity. The main insight gained by this exercise is that, under certain conditions, the savings in the lower bound cost resulting from a reduction in setup times increase as we further decrease the setup times.

APPENDIX

Proof of Proposition 2.

PROOF: To check whether the sum of two connected functions is connected, we need to find a continuous path from $\mathbf{x}$ to $\mathbf{y}$ such that value of $f + g$ along the path is never larger than the maximum of $f(\mathbf{x}) + g(\mathbf{x})$ and $f(\mathbf{y}) + g(\mathbf{y})$. Pick $\mathbf{x}$ and $\mathbf{y}$ arbitrarily. Without loss of generality we can and do assume that $f(\mathbf{x}) + g(\mathbf{x}) \geq f(\mathbf{y}) + g(\mathbf{y})$ and that $f(\mathbf{x}) \geq f(\mathbf{y})$.

There are two cases to consider, namely $g(\mathbf{x}) \geq g(\mathbf{y})$ and $g(\mathbf{x}) < g(\mathbf{y})$. Let us assume first that $g(\mathbf{x}) \geq g(\mathbf{y})$. Starting from $\mathbf{x}$ take any path in the set

$$\{\mathbf{z} : f(\mathbf{z}) = f(\mathbf{x}), g(\mathbf{y}) \leq g(\mathbf{z}) \leq g(\mathbf{x})\}$$

in the direction towards which $g(\cdot)$ decreases. Stop at the point where the level set $\{\mathbf{z} : f(\mathbf{z}) = f(\mathbf{x})\}$ intersects the set $\{\mathbf{z} : g(\mathbf{z}) = g(\mathbf{y})\}$ and move towards $\mathbf{y}$ within the set

$$\{\mathbf{z} : g(\mathbf{z}) = g(\mathbf{y}), f(\mathbf{z}) \leq f(\mathbf{x})\}.$$

Notice that in the first portion of the path we have $f(\mathbf{z}) + g(\mathbf{z}) \leq f(\mathbf{x}) + g(\mathbf{x})$ since $f(\mathbf{z}) = f(\mathbf{x})$ and $g(\mathbf{z}) \leq g(\mathbf{x})$.

In the second part of the path we have $f(\mathbf{z}) + g(\mathbf{y}) \leq f(\mathbf{x}) + g(\mathbf{x})$ since $f(\mathbf{z}) \leq f(\mathbf{x})$ and $g(\mathbf{y}) \leq g(\mathbf{x})$.

Now consider the case $g(\mathbf{x}) < g(\mathbf{y})$. Starting from $\mathbf{x}$ take any path in the set

$$\{\mathbf{z} : g(\mathbf{z}) = g(\mathbf{x}), f(\mathbf{y}) \leq f(\mathbf{z}) \leq f(\mathbf{x})\}$$

in the direction in which $f(\cdot)$ decreases. Stop at the point where the path intersects the set $\{\mathbf{z} : f(\mathbf{z}) = f(\mathbf{y})\}$ and move towards $\mathbf{y}$ within the set

$$\{\mathbf{z} : f(\mathbf{z}) = f(\mathbf{y}), g(\mathbf{z}) \leq g(\mathbf{y})\}.$$

Notice that in the first portion of the path we have $f(\mathbf{z}) + g(\mathbf{z}) \leq f(\mathbf{x}) + g(\mathbf{x})$ since $f(\mathbf{z}) \leq$

$f(\mathbf{x})$ and $g(\mathbf{z}) = g(\mathbf{x})$. In the second part of the path we have $f(\mathbf{z}) + g(\mathbf{z}) \leq f(\mathbf{x}) + g(\mathbf{x})$, since $f(\mathbf{z}) \leq f(\mathbf{y})$, $g(\mathbf{z}) \leq g(f)$, and $f(\mathbf{y}) + g(\mathbf{y}) \leq f(\mathbf{x}) + g(\mathbf{x})$. ■

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