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Optimal start time of a markdown sale under a two-echelon inventory system

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Abstract

The importance of inventory management for perishable items has been steadily attracting attention. Because of the characteristics of items whose values drop precipitously or cannot be sold after a particular time, items should be disposed of by a markdown sale. Accordingly, the company makes the following decisions at the end of the selling season: (a) selection on which products to be discounted, (b) pricing of the product, and (c) timing of the sale. Extant literature on the inventory problem has mainly focused on investigating decisions on selecting products for discount and the amount of the discount. That is, the decision on the start time of the markdown sale was not extensively studied. This study focuses on the optimal combination of a start time of the markdown sale and an order quantity based on a newsvendor model. Under certain conditions in a decentralized system, the start time of a markdown sale, where the retailer obtains the highest profit, is the least profitable for the manufacturer. Therefore, we propose a revenue-sharing contract to avoid irrational ordering behavior by a retailer against a manufacturer. Centralization through the revenue-sharing contract improves the profits of the retailer and manufacturer compared to those earned in the decentralized system.

Keywords: newsvendor model; markdown sale; revenue-sharing contract; supply chain coordination

1. Introduction

Inventory management on *perishable items* has been steadily attracting attention from researchers in various academic fields, including operation management, marketing, and business administration. In general, perishable items refer to products that see a precipitous drop in value or that cannot be sold after a certain time because of their finite or limited shelf life. In the past, the term was used to describe products, especially food, that decay quickly. In recent years, however, as product development life cycles have shortened and global competition has intensified, more types

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of products have come to be regarded as perishable items. For example, high-tech devices, such as mobile phones, are launched more often than ever before, and fast fashion goods that were used to be introduced quarterly are now released monthly or weekly. The lifespan of food in a supermarket also decreases because of an increase in customer demand for freshness (Nakandala et al., 2017). Accordingly, the traditional method running the inventory by maintaining stocks for a long period no longer confers a competitive advantage. Customers regard the products already stored in inventory for a long time as technically cluttered or stale products (Aviv and Pazgal, 2008). Consequently, keeping goods in stock over a long time eventually causes loss of profitability (Avinadav and Arponen, 2009).

Various studies have been widely conducted to deal with perishable items. A newsvendor model is one of the conventional approaches used to cope with perishable items in inventory management. The model provides an optimal order quantity by considering the trade-off between overestimating and underestimating customer demand. A general assumption in the basic newsvendor model is that a retailer orders a single item from a manufacturer (supplier) by determining the optimal order quantity to meet the uncertain demand within a single period. This classical model has been extended to various ways (Khouja, 1999; Qin et al., 2011). Although various extensions of the model were developed, a fluctuation of the price within a single period was not considered. Even though the newsvendor model was extended to multi-period model, it solved the problem recursively based on the single-period model. In the case of perishable item, it would be worth noting by expressing the price fluctuation during a single period to illustrate the last-order situation at the end of the selling season.

For perishable items near the end of the selling season, the company might earn more profit by selling all of the remaining stocks with the lower price rather than disposing off the entire leftover stocks. Outdated stocks not only hinder the flow of capital but also occupy the space used for a new product. In addition, relatively old products lose competitiveness because of the new entry of competitive products into the market. Furthermore, a company selling an out-of-style item at a low price can degrade the brand image. According to *The Times* magazine, *Burberry*, which is a luxury brand, incinerated up to as much of £90 million worth of stock in July 2018.

To deal with these issues, many companies have introduced *pricing strategy* to reduce the loss incurred by perishable items. It refers not to passive acceptance of existing customer demand but the proactive response for amplification of demand. By reducing the price for the same product over time, more demand can be generated by attracting interests from customers who want to purchase the product at a sale price. A markdown sale is a representative example of a pricing strategy. In the case of a promotion, the price is not permanently being reduced, and it can change over time. It is related to the studies considering dynamic pricing or multiple price markdowns (Gupta et al., 2006; Yang and Zhang, 2014; Chung et al., 2015; Moon et al., 2016). In the literature on economics, it is widely known that the price and demand are in inverse proportion to each other (You, 2005; Kocabıyıkoğlu and Popescu, 2011; Yang et al., 2013). Research on forward-looking customers, who are willing to wait for a price reduction and make a purchase when the price is discounted, also explains the inverse relationship of price and demand. Pesendorfer (2002) claimed that customers who put a low valuation on a product expect the product to be sold at a lower price in a markdown sale. By accommodating the expectations of these customers, especially in the apparel industry, the company promotes a markdown sale for overstock items at the end of the selling season.

The area of inventory management also has shown an interest in pricing (Elmaghraby and Keskinocak, 2003). In addition to research areas such as demand forecasting, optimal order quantity, or pricing, researchers in the inventory management have focused on determining the price and order quantity simultaneously (Smith and Achabal, 1998; Petruzzi and Dada, 1999; Panda et al., 2015; Mitra, 2018). By extending the newsvendor model including determination of the price of the product, the model provides the optimal price and order quantity at a time (Federgruen and Heching, 1999; Chen et al., 2011; Hu et al., 2015). However, another important issue has been overlooked: *the timing* of a markdown sale. It may bring about the following question: *What is the optimal time to reduce the price?* The determination of an appropriate start time of price reduction could remove the unnecessary inventory while maximizing the revenue. This information could also have a significant impact on the retailer's last-order quantity, which consequently affects the profit of the manufacturer and supply chain system. Depending on the situation, a retailer might earn the maximum profit when a markdown sale starts as early as possible. In contrast, a late markdown sale generates maximum profit. In a particular case, starting a sale in the middle of the selling period leads to the maximum profit. Otherwise, the retailer is indifferent to the start time of the markdown sale. Depending on the start time of the sale, customer demand and the order quantity of the retailer from the manufacturer could vary. In other words, the profits of the retailer and manufacturer vary based on the start time of the sale.

In this study, we analyzed the optimal combination of the start time of a markdown sale and an order quantity to generate the maximum profit at the end of the selling season. We extended the newsvendor model to consider the start time of a markdown by dividing the single period into two parts with (a) a regular price and (b) a sale price. In practice, the discount rate of a markdown sale, such as 30%, 40%, or 50%, is often predetermined. In contrast with many studies concentrated on the pricing and order quantity in the inventory model, we focus on an optimal combination of the start time of the sale and the order quantity under the predetermined discount rate.

When the retailer determines the optimal start time of the markdown sale and the order quantity from an individual perspective, it may lead to a local optimum. In other words, a decision made under a decentralized system cannot achieve maximum profit from the perspective of the overall supply chain system. In this system, the optimal quantity ordered by the retailer is different from the optimal quantity that the manufacturer would like to sell. Researchers have studied supply chain contracts in an effort to determine how to prevent such a local optimum. If the contract is appropriately designed, the supply chain is coordinated to ensure that the optimal order quantities for both retailer and manufacturer coincide. Naturally, total supply chain profit, which includes the profits of the retailer and manufacturer, can thus be maximized. In a similar manner, the start time of the markdown sale should also be considered from the perspective of supply chain coordination. The retailer and manufacturer may prefer different start times for the markdown sale under a decentralized system. Thus, the contract mechanism must be properly designed to achieve supply chain coordination in terms of the start time of the markdown sale. Therefore, we examine supply chain coordination after analyzing the decentralized system.

The remainder of the paper is organized as follows. Section 2 describes the demand modeling used for this research. Specifications of the profit functions and decisions of the retailer and manufacturer under a decentralized system are addressed in Section 3. Section 4 presents the profit functions and decisions under a centralized system through a revenue-sharing contract. The findings of this research are summarized in Section 5.

Table 1
Random variables representing uncertain demand considered in this study

$\xi = y(p) + \epsilon$	Demand in [0, T] when the item is sold at the regular price
$\xi' = y(p, \alpha) + \epsilon$	Demand in [0, T] when the item is sold at the sale price
$D = \frac{t_m}{T}\xi = \frac{t_m}{T}y(p) + \frac{t_m}{T}\epsilon$	Demand in [0, t_m] when the item is solved at the regular price
$D' = (\frac{T-t_m}{T})\xi' = \frac{T-t_m}{T}y(p, \alpha) + \frac{T-t_m}{T}\epsilon$	Demand in [0, t_m], demand in [t_m, T] when the item is solved at the sale price
$\epsilon \sim N(0, \sigma^2)$	Random variable following a normal distribution f as probability density function and F as a cumulative distribution function

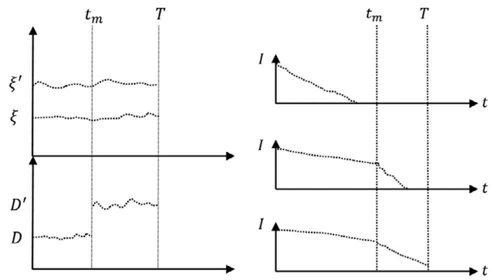

Fig. 1. Uncertain demand considered in this study and three types of inventory levels.

2. Problem description

In this study, we assume that a single retailer (newsvendor) places an order to a single manufacturer (supplier) at the end of the selling season. After observing the wholesale price and other relevant costs, the retailer determines the start time of the markdown sale and the order quantity. Let $t \in [0, T]$ denote the planning horizon where the markdown sale starts at $t = t_m$. The selling period ends at $t = T$, which is the expiration date for remaining items. The period is divided into two parts as $[0, t_m]$ and $[t_m, T]$. Until t_m, items are sold at a selling (regular) price, which is subsequently decreased with a discount rate $\alpha \in (0, 1)$ after t_m. Both price and discount rate were exogenously determined. The order is placed at $t = 0$ and covered until T. After T, no additional profit can be earned. Table 1 presents a summary of the random variables representing the uncertain demand considered in this study.

The terms $y(p)$ and $y(p, \alpha)$ represent the general *price-dependent* functions with the deterministic demand where the discount rate α serves to lower the price and increase the demand. These functions represent the expected demand in the planning horizon. We adopted the additive demand function, where $y(p) = a - bp$ and $y(p, \alpha) = a - (1 - \alpha)bp$. The notation ϵ incorporates a *price-independent* random variable that denotes the demand uncertainty. We assume that both random terms, indicated by ϵ in the ξ and ξ', are independent and identically distributed (IID). It should be noted that the random variables representing the demands are D and D', instead of using ξ and ξ'. D and D' represent the random variables following the uncertain demand in $[0, t_m]$ and $[t_m, T]$, respectively. The demands D and D' are expressed as a linear combination of ξ and ξ' with the ratio for each sale period in the planning horizon. Figure 1 illustrates the random variables representing the uncertain demand and the three possible situations for the inventory level in the planning horizon $[0, T]$. We assume that the total demand in the planning horizon is controllable by changing the start time of the markdown sale, t_m. If the selling period with the regular price becomes longer, t_m becomes larger in D and $(\frac{T-t_m}{T})$ becomes smaller in D'. That is, demand more increases if the period of price reduction is extended longer. Conversely, demand more decreases if the period of price reduction is shortened. To support that there is no major contradiction in the assumption, limits for t_m as to 0 and T are described in Equations (1) and (2), respectively:

$$\lim_{t_m \to T-} D = \lim_{t_m \to T-} \frac{t_m}{T}\xi = \lim_{t_m \to T-} \left(\frac{t_m}{T} y(p) + \frac{t_m}{T}\epsilon\right) = y(p) + \epsilon = \xi \tag{1}$$

$$\lim_{t_m \to 0+} D' = \lim_{t_m \to 0+} \left(\frac{T - t_m}{T}\right)\xi' = \lim_{t_m \to 0+} \left(\frac{T - t_m}{T} y(p, \alpha) + \frac{T - t_m}{T}\epsilon\right) = y(p, \alpha) + \epsilon = \xi'. \tag{2}$$

As shown in Equation (1), when t_m reaches T, which means the product is sold at the selling price p for the entire period $[0, T]$, the demand is equivalent to ξ. Similarly, by Equation (2), when t_m approaches 0, which means the product is sold at the sale price $(1 - \alpha)p$ for the entire period, the demand function becomes equal to ξ'. It is reasonable that the mean of each demand is proportional to the remaining duration for each sale in D and D', respectively. As shown in Table 1, the variances are also proportional to the duration of the sales in D and D'. Conceptually, ϵ expresses the uncertainty of the demand indicating that demand cannot be accurately predicted due to external or internal factors. In other words, the variance of the demand caused by uncertainty may increase when the remaining selling period is extended. Accordingly, the variance of demand is expressed as a product of the variance ϵ and the remaining selling period. For a general expression, ϵ can be used as a different random variable instead of an IID, but two reasons support the argument for setting it as an IID. First, by setting ϵ as an IID, the difference of variance can be affected solely by the remaining period rather than other factors. The main objective of our study is to analyze how the retailer's order quantity varies depending on the length of the remaining selling period. Therefore, we made the variance dependent only on the remaining period. Second, for ease of analysis, the two random variables are set to IID. Setting the random variables in different manners makes it challenging to deal with the expected profit function. Consequently, the analysis becomes difficult and the interpretation may not be intuitive. Therefore, we assume variances in demand functions as IID.

3. Analysis of the decentralized system

We analyzed a decentralized system in which a retailer and manufacturer consider the profit maximization from their respective positions. The retailer determines the start time of the markdown sale and the order quantity for the profit maximization within a given parameter. Meanwhile, the profit of the manufacturer depends on the order quantity determined by the retailer. Denote by q^* as the optimal order quantity when the start time of the markdown sale t_m is given. The term t_m^* indicates the optimal start time of the markdown sale when the order quantity q is given. The optimal combination for maximizing the expected profit function of the retailer is defined as (t_m^{**}, q^{**}). A newsvendor model is introduced to incorporate the expected profit function of the retailer.

3.1. Newsvendor model for a retailer

An objective function of a retailer is to maximize the total expected profit at the end of the selling season. Let $R_1(q, t_m)$, $R_2(q, t_m)$, and C be defined as

$$\begin{aligned} R_1(q, t_m) &= p \cdot \mathbb{E}[\min(q, D)] \\ R_2(q, t_m) &= (1-\alpha) \cdot p \cdot \mathbb{E}\big[\min\big((q-D)^+, D'\big)\big] \\ C &= c_r q + wq, \end{aligned}$$

where $\mathbb{E}$ denotes expectation and $(\mathbf{x})^+ = \max(\mathbf{x}, 0)$. The expected profit function of the retailer Π_r in the planning horizon $t \in [0, T]$ can be expressed as follows:

$$\begin{aligned} \Pi_r(q, t_m) &= R_1(q, t_m) + R_2(q, t_m) - C \\ &= p \cdot \mathbb{E}[\min(q, D)] + (1-\alpha) \cdot p \cdot \mathbb{E}\big[\min\big((q-D)^+, D'\big)\big] - c_r q - wq. \end{aligned}$$

The expected profit on the planning horizon is the difference between the sum of the two types of revenues in $[0, t_m]$ and $[t_m, T]$, and the total ordering cost. Decision variables q and t_m are defined as the order quantity and start time of the markdown sale, respectively, which are nonnegative real variables. Without loss of generality, the lead time is not taken into account, which means the order quantity q is held in stock at time $t = 0$. The revenue R_1 in $[0, t_m]$ is described by $p \cdot \mathbb{E}[\min(q, D)]$ for the product of the selling price p with the smaller value between the uncertain demand in $[0, t_m]$ and order quantity q. R_2 is the revenue in $[t_m, T]$ expressed as $(1-\alpha) \cdot p \cdot \mathbb{E}[\min((q-D)^+, D')]$ for the product of the sale price $(1-\alpha) \cdot p$ with the smaller value between the remaining inventories at t_m and the uncertain demand in $[t_m, T]$. The total ordering cost is expressed in $c_r q + wq$, where c_r is the retailer's per-unit cost and w is the wholesale price for a transfer payment. To avoid triviality, $c_r \in [0, p]$ was assumed.

In the existing literature on inventory management, although the salvage value was imposed on the leftover stock in common, it is not considered in this study. Although it is included in the profit function, it does not have a significant effect on the analysis. Therefore, the salvage value is not considered in this study.

Proposition 1. *The expected profit function of the retailer Π_r is strictly concave respect to q, where $q \geq 0$ and given $t_m \in [0, T]$. Therefore, there exists a unique q^* maximizing the expected profit function Π_r when t_m is given.*

Proof. See Appendix A. □

Proposition 2. *When an optimal order quantity $q^*(t_m)$ is given, critical ratio (fractile) $\frac{p-(c_r+w)}{p}$ can be expressed as a convex combination of $F(\frac{T}{t_m}q^* - (a-bp))$ and $F(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp)$:*

$$\alpha F\left(\frac{T}{t_m}q^* - (a-bp)\right) + (1-\alpha)F\left(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) = \frac{p-(c_r+w)}{p}.$$

Proof. See Appendix B. □

Corollary 1. *An optimal order quantity of the retailer has the lower and upper bounds shown in the following inequality:*

$$\frac{t_m}{T}F^{-1}\left(\frac{p-(c_r+w)}{p}\right)+\frac{t_m}{T}(a-bp)\le q^*\le F^{-1}\left(\frac{p-(c_r+w)}{p}\right)+a-bp+\frac{T-t_m}{T}\alpha bp. \tag{3}$$

Proof. By Proposition 1, the following inequality holds true:

$$F\left(q^*-a+bp-\frac{T-t_m}{T}\alpha bp\right)\le\frac{p-(c_r+w)}{p}\le F\left(\frac{T}{t_m}q^*-(a-bp)\right). \tag{4}$$

If Inequality (4) is rearranged based on q^*, it is equal to Inequality (3). □

Also, Inequality (4) can be expressed with the expectations of demands D and D' as shown in Inequality (5). Recall that $\mathbb{E}[D]=\frac{t_m}{T}(a-bp)$ and $\mathbb{E}[D+D']=a-bp+\frac{T-t_m}{T}\alpha bp$:

$$\frac{t_m}{T}F^{-1}\left(\frac{p-(c_r+w)}{p}\right)+\mathbb{E}[D]\le q^*\le F^{-1}\left(\frac{p-(c_r+w)}{p}\right)+\mathbb{E}[D+D]. \tag{5}$$

Proof. $\mathbb{E}[D+D']$ is larger than or equal to $\mathbb{E}[D]$. Also $F^{-1}(\frac{p-(c_r+w)}{p})$ is always larger than $\frac{t_m}{T}F^{-1}(\frac{p-(c_r+w)}{p})$ because $t_m\le T$ and $F(\cdot)$ is the nondecreasing function. □

Although a closed form is not proposed for q^*, lower and upper bounds of q^* are suggested. These bounds can be utilized to obtain q^* efficiently through the bisection method. Details are given in the next subsection. We will analyze how q^* varies with changes of t_m. Depending on t_m, the optimal order quantity q^* and the expected profit vary. Accordingly, we need to analyze the profit function with respect to t_m.

Proposition 3. *The expected profit function of the retailer Π_r is strictly concave with respect to t_m, where $t_m\in[0,T]$ and given $q\ge 0$. Thus, there exists a unique t_m^* maximizing the expected profit function Π_r when q is given.*

Proof. See Appendix C. □

Propositions 1 and 3 show that Π_r is strictly concave with respect to q and t_m. We now show that Π_r is jointly concave with q and t_m.

Proposition 4. *The expected profit function of the retailer Π_r is strictly concave with respect to q and t_m, where $t_m\in[0,T]$ and $q\ge 0$. There exists a unique combination (t_m^{**},q^{**}) maximizing the expected profit function of the retailer Π_r.*

Proof. See Appendix D. □

According to Proposition 4, the expected profit function is maximized through the optimal combination of the start time of the markdown sale and the order quantity. The solution procedure for obtaining the optimal combination is described in the next subsection.

3.2. Solution procedure for an optimal combination (t_m^{**}, q^{**})

We applied a bisection method to obtain the optimal combination of the start time of the markdown sale and the order quantity. The bisection method is one of the root-finding methods. It searches a solution by repeating a procedure based on dividing an initially given interval until the value of the function is less than tolerance (TOL). The criterion for dividing the interval is whether the value of the function obtained by the middle point of the interval is positive or negative. The procedure is repeated by dividing the given interval in half and defining each half as a new interval. When the value of the function of the middle point is smaller than the tolerance, the procedure is terminated. According to Bolzano's intermediate value theorem, the bisection method is guaranteed to converge if $h(a)$ and $h(b)$ have opposite signs, where $h(\cdot)$ is a continuous function in the interval $[a, b]$ (Russ, 1980). That is, q^* can be obtained by setting the value of the first derivative of the expected profit function to 0. The initial interval was set as the lower and upper bounds of the q^* in Corollary 1. Likewise, the optimal start time of the markdown sale t_m^* can be obtained using the initial interval $[0, T]$. In this manner, the optimal combination (t_m^{**}, q^{**}) can be obtained by iterating the procedure recursively. The pseudocode of the bisection method algorithm is described in Algorithm 1.

3.3. Profit function of a manufacturer

In this study, we assume that the capacity of the manufacturer is infinite. Under the assumption, the profit of the manufacturer is proportional to q determined by the retailer. The profit function of the manufacturer Π_m can be expressed as follows:

$$\Pi_m(q) = wq - c_m q,$$

where w and c_m represent the wholesale price and the manufacturer's per-unit cost, respectively. For the manufacturer, the lower and upper bounds of the order quantity by the retailer can be expected. Therefore, the lower and upper bounds of the expected profit of the manufacturer are as follows:

$$\text{Lower bound: } \left(w - c_m\right) \cdot \left(\frac{t_m}{T} F^{-1}\left(\frac{p - (c_r + w)}{p}\right) + \frac{t_m}{T}(a - bp)\right)$$

$$\text{Upper bound: } \left(w - c_m\right) \cdot \left(F^{-1}\left(\frac{p - (c_r + w)}{p}\right) + a - bp + \frac{T - t_m}{T}\alpha bp\right).$$

Although the order quantity is assumed as infinite, the profit of the manufacturer occurs within the interval given as described. When the wholesale price is fixed, the profit of the manufacturer varies with t_m as determined by the retailer. Also, the determination of t_m by the retailer depends on the wholesale price. Therefore, the profit of the manufacturer depends on the optimal combination of the start time of markdown sale and the order quantity determined by the retailer in the decentralized system. A detailed analysis was conducted with numerical experiments.

Algorithm 1. Bisection method algorithm

Initialization:
TOL=sufficiently small value
$LB = \frac{t_m}{T} F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + \frac{t_m}{T}(a-bp)$
$UB = F^{-1}\left(\frac{p-(c_r+w)}{p}\right) + (a-bp) + \frac{T-t_m}{T}\alpha bp$
$LT = TOL$
$UT = T$
$u(g, t_m) = p - \alpha pF\left(\frac{T}{t_m}q - (a-bp)\right) - (1-\alpha)pF\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) - (c_r + w)$
$v(g, t_m) = (a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right) - (1-\alpha)bpF\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) + \int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx$
while $|v(q, t_m)| \leq TOL$ **do**
  $t_m \leftarrow (LT + UT)/2$
  while $|u(q, t_m)| \leq TOL$ **do**
    $q \leftarrow (LB + UB)/2$
    if $u(LB, t_m) \cdot u(q, t_m) < 0$ **then**
      $UB \leftarrow q$
    end
    else
      $LB \leftarrow q$
    end
  end
  $q^* \leftarrow q$
  return $q^*, \Pi_r(q^*, tm)$
  $t_m^* \leftarrow \arg\max \Pi_r(q^*, t_m)$
  if $v(q^*, LT) \cdot v(q^*, t_m^*) < 0$ **then**
    $UT \leftarrow t_m^*$
  end
  else
    $LT \leftarrow t_m^*$
  end
end
$q^{**} \leftarrow q^*$
$t_m^{**} \leftarrow t_m^*$

3.4. Numerical experiments of the decentralized system

We conducted numerical experiments to analyze the decentralized system. The parameter setting for the experiments is provided in Table 2. We analyzed the optimal combination of the start time of the markdown sale and the order quantity determined by the retailer and how it affects the profits of the manufacturer and the supply chain system. We considered the following questions for the numerical experiments:

Table 2
Parameter setting for the numerical experiments of a decentralized system

	p	a	b	α	w	c_r	c_m
Case 1	120	7000	50	0.4	35	20	24
Case 2	120	7000	45	0.4	35	20	24
Case 3	120	6200	45	0.4	20	15	17
Case 4	120	7500	35	0.4	20	5	8

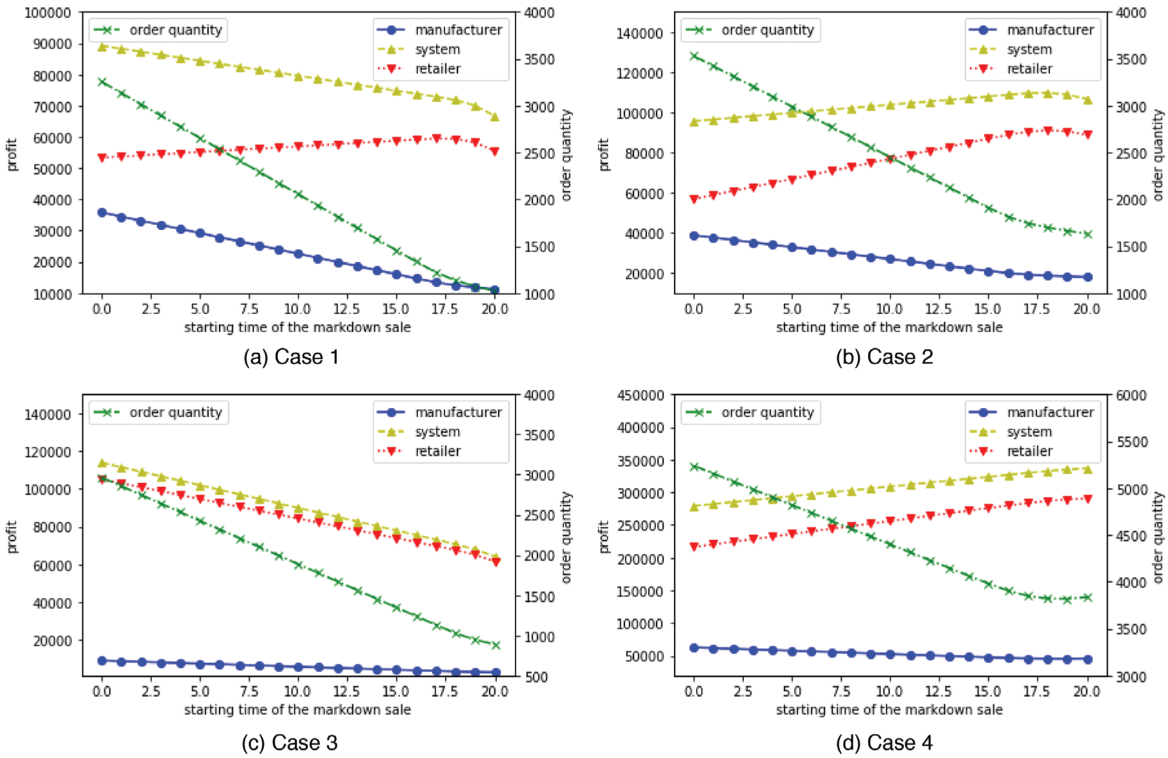

Fig. 2. Numerical experiments of the decentralized system.

(i) How does the optimal order quantity vary with the change of the start time of the markdown sale?
(ii) What is the start time of the markdown sale generating the maximum profit for the retailer?
(iii) How does the profit of the manufacturer vary?
(iv) How does the profit of the system vary?

We also solved the problem by varying the given start time of the markdown sale t_m to confirm whether the optimal combination proposed in this study actually generates the maximum profit. That is, the optimal order quantity and the expected profits of the retailer, manufacturer, and system (Π_r, Π_m, and Π_s, respectively) were estimated by fixing t_m. The results of the numerical experiments are illustrated in Fig. 2 and detailed results are provided in Table 3. The optimal combination

Table 3
Numerical experiments for a decentralized system

t	Case 1 q^*	Π_r	Π_m	Π_s	Case 2 q^*	Π_r	Π_m	Π_s
0	3256	53,363	35,816	89,179	3530	56,821	38,830	95,651
1	3136	53,723	34,496	88,219	3422	58,825	37,642	96,467
2	3016	54,083	33,176	87,259	3314	60,829	36,454	97,283
3	2896	54,443	31,856	86,299	3206	62,833	35,266	98,099
4	2776	54,803	30,536	85,339	3098	64,837	34,078	98,915
5	2656	55,163	29,216	84,379	2990	66,841	32,890	99,731
6	2536	55,523	27,896	83,419	2882	68,845	31,702	100,547
7	2416	55,883	26,576	82,459	2774	70,849	30,514	101,363
8	2296	56,243	25,256	81,499	2666	72,853	29,326	102,179
9	2176	56,603	23,936	80,539	2558	74,857	28,138	102,995
10	2056	56,963	22,616	79,579	2450	76,861	26,950	103,811
11	1936	57,323	21,296	78,619	2342	78,865	25,762	104,627
12	1816	57,683	19,976	77,659	2234	80,869	24,574	105,443
13	1696	58,043	18,656	76,699	2126	82,873	23,386	106,259
14	1576	58,403	17,336	75,739	2018	84,877	22,198	107,075
15	1456	58,763	16,016	74,779	1911	86,876	21,021	107,897
16	1336	59,123	14,696	73,819	1815	88,798	19,965	108,763
17	1222	59,440	13,442	72,882	1745	90,333	19,195	109,528
18	1135	59,367	12,485	71,852	1700	91,026	18,700	109,726
19	1073	58,207	11,803	70,010	1665	90,547	18,315	108,862
20	1021	55,478	11,231	66,709	1633	88,764	17,963	106,727

t	Case 3 q^*	Π_r	Π_m	Π_s	Case 4 q^*	Π_r	Π_m	Π_s
0	2966	104,927	8898	113,825	5239	216,510	62,868	279,378
1	2858	102,851	8574	111,425	5155	220,482	61,860	282,342
2	2750	100,775	8250	109,025	5071	224,454	60,852	285,306
3	2642	98,699	7926	106,625	4987	228,426	59,844	288,270
4	2534	96,623	7602	104,225	4903	232,398	58,836	291,234
5	2426	94,547	7278	101,825	4819	236,370	57,828	294,198
6	2318	92,471	6954	99,425	4735	240,342	56,820	297,162
7	2210	90,395	6630	97,025	4651	244,314	55,812	300,126
8	2102	88,319	6306	94,625	4567	248,286	54,804	303,090
9	1994	86,243	5982	92,225	4483	252,258	53,796	306,054
10	1886	84,167	5658	89,825	4399	256,230	52,788	309,018
11	1778	82,091	5334	87,425	4315	260,202	51,780	311,982
12	1670	80,015	5010	85,025	4231	264,174	50,772	314,946
13	1562	77,939	4686	82,625	4147	268,146	49,764	317,910
14	1454	75,863	4362	80,225	4063	272,118	48,756	320,874
15	1346	73,787	4038	77,825	3981	276,082	47,772	323,854
16	1238	71,711	3714	75,425	3905	279,990	46,860	326,850
17	1130	69,634	3390	73,024	3849	283,676	46,188	329,864
18	1026	67,515	3078	70,593	3821	286,860	45,852	332,712
19	944	65,000	2832	67,832	3818	289,278	45,816	335,094
20	888	61,410	2664	64,074	3836	290,781	46,032	336,813

Table 4
Optimal combinations of the start time of the markdown sale and order quantity, and the profits of the retailer, manufacturer, and system from the numerical experiments in the decentralized system

	Case 1	Case 2	Case 3	Case 4
Start time of markdown sale	$t = 17.44$	$t = 18.12$	$t = 0.00$	$t = 20.00$
Order quantity	1179	1695	2966	3836
Profit of the retailer	59,496	91,034	104,927	290,781
Profit of the manufacturer	12,969	18,654	8,898	46,032
Profit of the system	72,465	109,688	113,825	336,813

of the start time of the markdown sale and the order quantity, and the profits of the retailer, manufacturer, and system, are described in Table 4. The answers to questions (i)–(iv) are presented in Observations 1–4.

Observation 1. *As shown in Fig. 2, when the start time of the markdown sale t_m increased, the optimal order quantity q^* of the retailer decreased.*

Because the probability distribution of the aggregated demand during the planning horizon follows $N(a - bp + \frac{(T-t_m)}{T}\alpha bp, \sigma^2)$, when t_m becomes larger, the lower demand occurs. On the contrary, when t_m approaches 0, the total demand of the customer increases. As shown in Inequality (3) in Corollary 1, when t_m approaches T, the range of the lower and upper bounds of q^* is close to $F^{-1}(\frac{p-c}{p}) + a - bp$, while the range is widened when t_m approaches 0. If the sale price $(1-\alpha)\cdot p$ is not less than the purchase cost $c_r + w$, the retailer places more order than $F^{-1}(\frac{p-c}{p}) + a - bp$ to cover the demand, which is larger than the demand ξ. Consequently, the optimal order quantity q^* for the retailer tends to increase with decreasing t_m. Thus, the inverse property between the optimal order quantity of the retailer and the start time of the markdown sale t_m was observed.

Observation 2. *As shown in Case 3, the maximum profit of the retailer was generated when the markdown sale started at $t_m = 0$. On the contrary, in Case 4, the maximum profit of the retailer was reached when the markdown sale started at $t_m = T$. In Cases 1 and 2, the retailer would choose the optimal start time of the markdown sale t_m^* as 17.44 and 18.12, respectively.*

Corollary 2. *If the wholesale price w is set to be less than $-c_r + 2p - \alpha p - \frac{a}{b}$, it is a sufficient condition for the retailer starting the markdown sale at $t_m = 0$ where the manufacturer makes the relatively larger profit compared to the opposite case.*

Proof. Since the expected profit function of the retailer is concave with respect to t_m, when the first derivative of the function has negative value at $t_m = 0$, it also has a negative value even after $t_m = 0$. Therefore, when the wholesale price w is set to be less than $-c_r + 2p - \alpha p - \frac{a}{b}$, the retailer acquires the maximum expected profit when the markdown sale starts at $t_m = 0$ (see Appendix E). At this time, the optimal order quantity of the retailer is as follows:

$$\lim_{t_m \to 0+} \frac{\partial \Pi_r(q, t_m)}{\partial q} = (1-\alpha)\cdot p \cdot (1 - F(q - (a - bp) + \alpha bp)) - \left(c_r + w\right).$$

Accordingly,

$$\lim_{t_m \to 0+} q^* = F^{-1}\left(\frac{(1-\alpha)\cdot p - (c_r + w)}{(1-\alpha)\cdot p}\right) + a - bp + \alpha bp. \qquad \square$$

Corollary 3. *If the following inequality is satisfied, the maximum profit of the retailer occurs when the markdown sale starts at $t_m = T$:*

$$(a - bp + \alpha bp)\left(\frac{p - (c_r + w)}{p}\right) - b \cdot (p - (c_r + w)) + \int_{-\infty}^{\frac{p-(c_r+w)}{p}} x f(x) dx > 0.$$

Proof. The proof process of Corollary 3 is similar to that of Corollary 2. Since the expected profit function Π_r is strictly concave with respect to t_m, if the value of the first derivative of the function is positive at $t_m = T$, then the maximum profit is reached at this point (see Appendix F). Thus, the optimal order quantity of the retailer is as follows:

$$\lim_{t_m \to T-} \frac{\partial \Pi_r(q, t_m)}{\partial q} = p \cdot (1 - F(q - (a - bp) + \alpha bp)) - (c_r + w).$$

Accordingly,

$$\lim_{t_m \to T-} q^* = F^{-1}\left(\frac{p - (c_r + w)}{p}\right) + a - bp. \qquad \square$$

Observation 3. *According to Observation 1, the optimal order quantity q^* by the retailer decreased as t_m increased. Because the profit function of the manufacturer follows $\Pi_m = wq - c_m q$, it is proportional to the order quantity of the retailer. Therefore, the manufacturer prefers that the retailer starts the markdown sale at $t_m = 0$.*

Customer demand is assumed to be controllable according to the start time of the markdown sale. From the perspective of the manufacturer, it is profitable when the retailer orders as many as possible. Therefore, the manufacturer prefers the start time of the markdown sale at $t_m = 0$, which amplifies the customer demand.

Observation 4. *The overall profit of the system also depends on the retailer's start time of the markdown sale. For example, in Case 1, which is illustrated in Fig. 2a, the system profit reached the maximum value when t_m was determined at $t_m = 0$, but the retailer benefited from starting the markdown sale at another time.*

In Cases 1 and 2, the most profitable start time of the markdown sale for the retailer differed from that of the manufacturer and the overall system. Especially for Case 1, for the manufacturer or system, the maximum profit was gained when the markdown sale started at $t = 0$, but the retailer determined the start time at $t = 17.44$. Meanwhile, for Cases 3 and 4, the retailer determined the start time of markdown sale, which also generated the maximum profits of the manufacturer and system despite the decentralized system. Although the optimal start time of the markdown sale of the retailer coincided with that of the manufacturer, the supply chain was not coordinated. It is necessary to analyze the optimal combination of a start time of the markdown sale and order quantity from a system point of view. Also, an appropriate distribution of the maximum system profit is required. The supply chain coordination based on the centralized system is discussed in the next section.

4. Analysis of a centralized system

In this section, an optimal combination (t_m^{**}, q^{**}) from the system perspective are considered. We adopted a revenue-sharing contract rather than a buy-back contract because the newsvendor model does not consider leftover stock. The main purpose of the contract is to change the profit structure to reach the *Pareto optimum*. Based on the *Stackelberg game*, the manufacturer who is the leader determines the contract parameters and the retailer who is the follower subsequently decides on a start time of the markdown sale and an order quantity. Under the revenue-sharing contract, the transfer payment from a retailer to a manufacturer includes a certain fraction of the retailer's revenue ℓ and the wholesale price w. The determination of the appropriate wholesale price proposed by Cachon and Lariviere (2005) is modified and the sufficient condition for this model is proposed for the revenue-sharing contract.

In the case of the decentralized system, because w is larger than c_m, the optimal order quantity from the viewpoint of the system is not placed. To establish the revenue-sharing contract, the manufacturer must provide the retailer with a wholesale price w that is less than c_m and receive a certain percentage of the revenue ℓ from the retailer.

4.1. Revenue-sharing contract

We suggest a revenue-sharing contract to overcome the relatively small profit of the system due to the decision from each standpoint. Under the revenue-sharing contract, the expected profit functions of the retailer and manufacturer, respectively, are as follows:

$$\Pi_r(q, t_m) = R_1(q, t_m) + R_2(q, t_m) - (w + c_r) \cdot q - (1 - \ell) \cdot (R_1(q, t_m) + R_2(q, t_m)) \tag{6}$$

$$\Pi_m(q, t_m) = (1 - \ell) \cdot (R_1(q, t_m) + R_2(q, t_m)) + wq - c_m q. \tag{7}$$

The total expected profit function of the supply chain system Π_s is as follows:

$$\Pi_s(q, t_m) = R_1(q, t_m) + R_2(q, t_m) - (c_r + c_m) \cdot q. \tag{8}$$

From the system perspective, the optimal combination (t_m^{**}, q^{**}) can be obtained through the Algorithm proposed in Section 3 by replacing the wholesale price w with the manufacturer's per-unit cost c_m. Denote q_r^*, q_m^*, and q_s^* by the optimal order quantities for the retailer, manufacturer, and system, respectively. If a certain fraction of the retailer's revenue ℓ and wholesale price w satisfy Equations (9) and (10), then $q_r^* = q_m^* = q_s^*$ holds true, which means the supply chain is coordinated:

$$\ell = \frac{w + c_r}{c_r + c_m} \tag{9}$$

$$w = -c_r + (c_r + c_m) \cdot \ell. \tag{10}$$

Corollary 4. *If the following Inequality (11) is satisfied, then the maximum profit of the system occurs when the retailer starts the markdown sale at* $t_m = 0$*:*

$$a - bp + \alpha bp - b \cdot (p - c_r - c_m) < 0. \tag{11}$$

Proof. It can be easily proved by referring to the proof of the Corollary 2 by replacing the wholesale price w with the manufacturer's cost per-unit c_m. □

The optimal order quantity by the retailer, manufacturer, and system, respectively, are as follows:

$$q_r^* = F^{-1}\left(\frac{\ell \cdot (1-\alpha) \cdot p - (w + c_r)}{\ell \cdot (1-\alpha) \cdot p}\right) + a - bp + \alpha bp$$

$$q_m^* = F^{-1}\left(\frac{(1-\ell) \cdot (1-\alpha) \cdot p - (c_m - w)}{(1-\ell) \cdot (1-\alpha) \cdot p}\right) + a - bp + \alpha bp$$

$$q_s^* = F^{-1}\left(\frac{(1-\alpha) \cdot p - (c_r + c_m)}{(1-\alpha) \cdot p}\right) + a - bp + \alpha bp.$$

When Equations (9) and (10) are satisfied, then $q_r^* = q_m^* = q_s^*$ holds true and the supply chain is coordinated.

Corollary 5. *If the following Inequality (12) is satisfied, then the maximum profit of the system occurs when the retailer starts the markdown sale at* $t_m = T$*:*

$$(a - bp + \alpha bp)\left(\frac{p - (c_r + w)}{p}\right) - b \cdot (p - c_r - c_m) + \int_{-\infty}^{\frac{p-(c_r+c_m)}{p}} x f(x) dx > 0. \tag{12}$$

Proof. The proof of Corollary 5 can be easily completed by replacing the wholesale price with the manufacturer's cost per unit in the proof of Corollary 3. □

In this case, the optimal order quantity by the retailer, manufacturer, and system, respectively, are as follows:

$$q_r^* = F^{-1}\left(\frac{\ell \cdot p - (w + c_r)}{\ell \cdot p}\right) + a - bp$$

$$q_m^* = F^{-1}\left(\frac{(1-\ell) \cdot p - (c_m - w)}{(1-\ell) \cdot p}\right) + a - bp$$

$$q_s^* = F^{-1}\left(\frac{(p - (c_r + c_m)}{p}\right) + a - bp.$$

When Equations (9) and (10) hold true, the supply chain is coordinated with $q_r^* = q_m^* = q_s^*$. Otherwise, the optimal combination of the start time of the markdown sale and order quantity can be obtained by the Algorithm using c_m instead of w. Under the coordination, the optimal combination

Table 5
Optimal combinations of the start time of the markdown sale and order quantity, and the profits of the retailer, manufacturer, and system from the numerical experiments in the centralized system

	Case 1 ($\ell = 0.75$)	Case 2 ($\ell = 0.83$)	Case 3 ($\ell = 0.92$)	Case 4 ($\ell = 0.86$)
Start time of markdown sale	$t = 0.00$	$t = 17.62$	$t = 0.00$	$t = 20.00$
Order quantity	3269	1809	2982	4115
Profit of the retailer	67,260	91,523	104,946	290,998
Profit of the manufacturer	22,420	18,746	8903	47,372
Profit of the system	89,680	110,269	113,849	338,370

(t_m^{**}, q^{**}) from the system is also the optimal combination for the retailer and manufacturer. By inserting the optimal combination (t_m^{**}, q^{**}) into the profit functions (6), (7), and (8), the maximum expected profits of the retailer, manufacturer, and system, respectively, are obtained.

4.2. Numerical experiments of the centralized system

Numerical experiments were conducted to characterize the centralized system. The parameter setting from Table 2 was also used for the experiments, except for the wholesale price w and a certain fraction of the retailer's revenue ℓ. A summary of the numerical experiment is provided in Table 5.

As can be seen from Table 5, all expected profits were higher than those shown in Table 4. The optimal order quantities by the retailer, manufacturer, and system were equal, showing that the supply chain was coordinated. Although setting w as less than c_m led to the revenue of the manufacturer as a negative value, the profit of the manufacturer was higher than the decentralized system. Notably, for Cases 1, 2, and 4, the optimal order quantities increased through coordination, but order quantity decreased in Case 3. In this case, the wholesale price w was set to be relatively small according to Corollary 2. For the retailer, the total purchasing cost was reduced, which resulted in placing a larger order quantity. Table 5 shows that the optimal start time of the markdown sale was consistent with the decentralized system in Cases 3 and 4. In Cases 1 and 2, however, the optimal start time of the markdown sale changed with the coordination. That is, when the start time of markdown sale is considered, supply chain coordination is required to match the optimal start time of markdown sale as well as the optimal order quantity.

5. Conclusions

As the inventory management for perishable items becomes critical to a company, it is necessary to adjust the customer demand by taking the start time of the markdown sale into consideration at the end of the selling season. In preparation for the end of the selling season, the retailer should consider not only the order quantity but also the start time of the markdown sale. However, it can be disadvantageous to the manufacturer because the profit depends on the decision of the retailer. That is, the decision in each perspective cannot reach the system's maximum profit. Using a revenue-sharing contract based on the appropriate fraction of the retailer's revenue and wholesale price, the maximum profit from the perspective of the system can be obtained.

5.1. *Managerial insights*

Research on the newsvendor model and supply chain coordination has been widely conducted. Also, variations of the model with regard to pricing have been widely developed. To the best of our knowledge, however, research considering the start time of a markdown sale is scarce. Our study produced the following managerial insights:

(i) When a retailer sets the start time of the markdown sale individually, it can result in being quite disadvantageous to the manufacturer. The profit of the manufacturer depends on the order quantity of the retailer when the wholesale price is fixed. Therefore, the manufacturer prefers that the retailer starts the markdown sale as soon as possible to amplify customer demand. However, the retailer determines the optimal order quantity depending on the given external factors; that is, it is difficult to match each preference of a start time of the markdown sale in the decentralized system.

(ii) When the manufacturer sets the wholesale price appropriately, the preferred start time of the markdown sale from the retailer and manufacturer can coincide. In this case, the profit difference between the decentralized and centralized system is relatively small, but the supply chain is not coordinated.

(iii) In general, a supply chain coordination based on the newsvendor model means that the optimal order quantity from the system point of view equals that of the retailer and manufacturer while the system profit is maximized. However, when the concept of determining the optimal start time of the markdown sale is introduced, not only matching the order quantity but also the start time must be considered. In other words, the supply chain coordination is achieved when the optimal combination (t_m^{**}, q^{**}) is realized.

5.2. *Directions for future study*

We investigated the insights from the newsvendor model by adding the notion of a start time of the markdown sale in both decentralized and centralized systems. Because the research of this concept is still insufficient, plenty of variations can be considered through this study. In terms of demand modeling, several methodologies could lead to practical results. In addition, because this study is extended based on the single-period newsvendor model, new insights can be found by extending the model to the multi-period, multi-item, multi-retailer, or multi-echelon model. This study is expected to be a cornerstone for numerous upcoming future studies.

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Appendix A

$$
\begin{aligned}
&\mathbb{E}[\min(q, D)] \\
&= \mathbb{E}[D|D \leq q] \cdot Pr(D \leq q) + \mathbb{E}[q|D \geq q] \cdot Pr(D \geq q) \\
&= \mathbb{E}\Big[\frac{t_m}{T}(a - bp) + \frac{t_m}{T}\epsilon | \frac{t_m}{T}(a - bp) + \frac{t_m}{T}\epsilon \leq q\Big] \cdot Pr\Big(\frac{t_m}{T}(a - bp) + \frac{t_m}{T}\epsilon \leq q\Big) \\
&\quad + \mathbb{E}\Big[q| \frac{t_m}{T}(a - bp) + \frac{t_m}{T}\epsilon \geq q\Big] \cdot Pr\Big(\frac{t_m}{T}(a - bp) + \frac{t_m}{T}\epsilon \geq q\Big) \\
&= \frac{t_m}{T} \cdot (a - bp) \cdot F\left(\frac{T}{t_m}q - (a - bp)\right) + \frac{t_m}{T} \cdot \mathbb{E}\left[\epsilon | \epsilon \leq \frac{T}{t_m}q - (a - bp)\right] \\
&\quad \cdot Pr\left(\epsilon \leq \frac{T}{t_m}q - (a - bp)\right) + q \cdot Pr\left(\epsilon \geq \frac{T}{t_m}q - (a - bp)\right) \\
&= \frac{t_m}{T}(a - bp) \cdot F\left(\frac{T}{t_m}q - (a - bp)\right) + \frac{t_m}{T} \cdot \frac{\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx}{Pr\left(\epsilon \leq \frac{T}{t_m}q - (a - bp)\right)} \\
&\quad + q \cdot \left[1 - F\left(\frac{T}{t_m}q - (a - bp)\right)\right] = \frac{t_m}{T} \cdot (a - bp)F\left(\frac{T}{t_m}q - (a - bp)\right) \\
&\quad + \frac{t_m}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx + q \cdot \left[1 - F\left(\frac{T}{t_m}q - (a - bp)\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\big[\min[\max(q - D, 0), D']\big] \\
&= Pr(D \leq q \leq D + D') \cdot (q - \mathbb{E}[D|D \leq q \leq D + D']) + Pr(D + D' \leq q) \\
&\quad \cdot \mathbb{E}[D' \mid D + D' \leq q] \\
&= Pr\left(\frac{t_m}{T}(a - bp) + \frac{t_m}{T}\epsilon \leq q \leq a - bp + \epsilon + \frac{T - t_m}{T}\alpha bp\right) \\
&\quad \cdot \left(q - \mathbb{E}\left[\frac{t_m}{T}(a - (1 - \alpha) \cdot bp) + \frac{T - t_m}{T}\epsilon \mid a - bp + \epsilon + \frac{T - t_m}{T}\alpha bp\right]\right) \\
&\quad + Pr\left(a - bp + \epsilon + \frac{T - t_m}{T}\alpha bp \leq q\right) \\
&\quad \cdot \mathbb{E}\left[\frac{T - t_m}{T}(a - (1 - \alpha) \cdot bp) + \frac{T - t_m}{T}\epsilon \mid a - bp + \epsilon + \frac{T - t_m}{T}\alpha bp \leq q\right],
\end{aligned}
$$

where

$$
Pr\left(\frac{t_m}{T}(a - bp) + \frac{t_m}{T}\epsilon \leq q \leq a - bp + \epsilon + \frac{T - t_m}{T}\alpha bp\right)
$$

$$
\begin{aligned}
&\cdot\left(q - \mathbb{E}\left[\frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon \mid \frac{t_m}{T}(a-bp) + \frac{t_m}{T}\epsilon \le q \le a - bp + \epsilon + \frac{T-t_m}{T}\alpha bp\right]\right)\\
&\quad = Pr\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp \le \epsilon \le \frac{T}{t_m}q - (a-bp)\right)\cdot\left(q - \frac{t_m}{T}(a-bp)\right)\\
&\qquad - \frac{t_m}{T}\int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx\\
&\quad = \left(q - \frac{t_m}{T}(a-bp)\right)\cdot\left(F\left(\frac{T}{t_m}q - (a-bp)\right) - F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)\right)\\
&\qquad - \frac{t_m}{T}\int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx
\end{aligned}
$$

and

$$
\begin{aligned}
&Pr\left(a - bp + \epsilon + \frac{T-t_m}{T}\alpha bp\right)\\
&\cdot\mathbb{E}\left[\frac{T-t_m}{T}(a-(1-\alpha)bp) + \frac{T-t_m}{T}\epsilon | a - bp + \epsilon + \frac{T-t_m}{T}\alpha bp \le q\right]\\
&\quad = F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)\\
&\qquad \cdot\left(\frac{T-t_m}{T}(a-(1-\alpha)bp) + \frac{T-t_m}{T}\int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx\right.\\
&\qquad \left.\cdot\frac{1}{F\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right)}\right)\\
&\quad = \frac{T-t_m}{T}(a - bp + \alpha bp)F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)\\
&\qquad + \frac{T-t_m}{T}\int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\left[\min[\max(q-D,0),D']\right]\\
&\quad = \left(q - \frac{t_m}{T}(a-bp)\right)F\left(\frac{T}{t_m}q - (a-bp)\right)\\
&\qquad - \left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)
\end{aligned}
$$

$$- \frac{t_m}{T} \int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx + \frac{T-t_m}{T} \int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx.$$

Accordingly,

$$\begin{aligned}
\Pi_r(q, t_m) &= p \cdot \mathbb{E}[\min(q, D)] + (1-\alpha)p \cdot \mathbb{E}\big[\min((q-D)^+, D')\big] - (c_r + w)q \\
&= p\frac{t_m}{T}(a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right) + p\frac{t_m}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx \\
&+ pq\left(1 - F\left(\frac{T}{t_m}q - (a-bp)\right)\right) + (1-\alpha)\left[p\left(q - \frac{t_m}{T}(a-bp)\right)F\left(\frac{t_m}{T}q - (a-bp)\right)\right. \\
&- \left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) \cdot F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) \\
&\left. - \frac{t_m}{T}\int_{q-(a-bp)-\frac{T-t_m}{T}\alpha bp}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx + \frac{T-t_m}{T}\int_{-\infty}^{q-(a-bp)-\frac{T-t_m}{T}\alpha bp} xf(x)dx\right] \\
&- (c_r + w)q.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \Pi_r(q, t_m)}{\partial q} &= p(a-bp)f\left(\frac{T}{t_m}q - (a-bp)\right) + p\left(\frac{T}{t_m}q - (a-bp)\right) + p\left(1 - F\left(\frac{T}{t_m}q - (a-bp)\right)\right) \\
&- pq\frac{T}{t_m}f\left(\frac{T}{t_m}q - (a-bp)\right) + (1-\alpha)p\left(F\left(\frac{T}{t_m}q - (a-bp)\right)\right) + \left(\frac{T}{t_m}q - (a-bp)\right) \\
&\cdot f\left(\frac{T}{t_m}q - (a-bp)\right) - F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) - \left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) \\
&\cdot f\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) - \left(\frac{T}{t_m}q - (a-bp)\right)f\left(\frac{T}{t_m}q - (a-bp)\right) \\
&+ \frac{t_m}{T}\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)f\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) \\
&+ \frac{T-t_m}{T}\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)f\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) - (c_r + w) \\
&= p\left(1 - F\left(\frac{T}{t_m}q - (a-bp)\right)\right) \\
&+ (1-\alpha)p\left(F\left(\frac{T}{t_m}q - (a-bp)\right) - F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)\right) - (c_r + w)
\end{aligned}$$

$$\frac{\partial^2 \Pi_r(q, t_m)}{\partial q^2} = -\frac{T}{t_m}\alpha pf\left(\frac{T}{t_m}q - (a-bp)\right) - (1-\alpha)pf\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) < 0.$$

$\because$ The probability density function $f(\cdot)$ is always nonnegative.

Appendix B

$$
\begin{aligned}
\frac{\partial \Pi_r(q,t_m)}{\partial q} &= p\left(1 - F\left(\frac{T}{t_m}q - (a-bp)\right)\right) \\
&+(1-\alpha)p\left(F\left(\frac{T}{t_m}q - (a-bp)\right) - F\left(q - (a-bp) - \frac{T-t_m}{T}\alpha bp\right)\right) - (c_r + w) \\
&\iff \alpha F\left(\frac{T}{t_m}q^* - (a-bp)\right) + (1-\alpha)F\left(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp\right) = \frac{p-(c_r+w)}{p}.
\end{aligned}
$$

Because the range of $F(\cdot)$ is between 0 and 1 and its value varies according to the q^*, $\frac{p-(c_r+w)}{p}$ can be expressed as a convex combination of $F(\frac{T}{t_m}q^* - (a-bp))$ and $F(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp)$. Thus, there exists a unique q^* maximizing the expected profit function of the retailer Π_r. Also, $F(\frac{T}{t_m}q^* - (a-bp))$ is always larger than $F(q^* - (a-bp) - \frac{T-t_m}{T}\alpha bp)$ because $\frac{T}{t_m}q - (a-bp) \geq q - (a-bp) - \frac{T-t_m}{T}\alpha bp$ holds true and $F(\cdot)$ has the nondecreasing property.

Appendix C

$$
\begin{aligned}
\frac{\partial \Pi_r(q,t_m)}{\partial t_m} &= \frac{p}{T}(a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right) + \frac{p}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx \\
&+ (1-\alpha)p\left[-\frac{1}{T}(a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right) - \frac{\alpha bp}{T}F\left(q-(a-bp) - \frac{T-t_m}{T}\alpha bp\right)\right. \\
&\left. - \frac{1}{T}\int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx\right] = \frac{\alpha p}{T}\left[(a-bp)F\left(\frac{T}{t_m}q - (a-bp)\right)\right. \\
&\left. - (1-\alpha)bpF\left(q-(a-bp) - \frac{T-t_m}{T}\alpha bp\right) + \int_{-\infty}^{\frac{T}{t_m}q-(a-bp)} xf(x)dx\right].
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \Pi_r(q,t_m)}{\partial t_m^2} &= \frac{\alpha p}{T}\left[-\frac{qT}{t_m^2}(a-bp)f\left(\frac{T}{t_m}q - (a-bp)\right) - \frac{\alpha b^2 p^2}{T}(1-\alpha)\right. \\
&\left. \cdot f\left(q-(a-bp) - \frac{T-t_m}{t_m}\alpha bp\right) - \frac{qT}{t_m^2}\left(\frac{T}{t_m}q - (a-bp)\right)f\left(\frac{T}{t_m}q-(a-bp)\right)\right] \\
&= -\frac{\alpha p q^2 T}{t_m^3}f\left(\frac{T}{t_m}q-(a-bp)\right) - \left(\frac{\alpha bp}{T}\right)^2(1-\alpha)pf\left(q-(a-bp)-\frac{T-t_m}{T}\alpha bp\right) < 0.
\end{aligned}
$$

Appendix D

Let $\mathbb{H}$ be a Hessian matrix of $\Pi_r(q, t_m)$ as follows: $\mathbb{H} = \begin{bmatrix} \frac{\partial^2 \Pi_r(q,t_m)}{\partial q^2} & \frac{\partial^2 \Pi_r(q,t_m)}{\partial q \partial t_m} \\ \frac{\partial^2 \Pi_r(q,t_m)}{\partial t_m \partial q} & \frac{\partial^2 \Pi_r(q,t_m)}{\partial t_m^2} \end{bmatrix}$. Using the second partial derivative test, it is easy to show that the determinant of the Hessian matrix, $\mathbb{D}$, is larger than 0:

$$\mathbb{D} = \left[-\frac{T}{t_m} \alpha p f\left(\frac{T}{t_m} q - (a - bp) \right) - (1 - \alpha) p f\left(q - (a - bp) - \frac{T - t_m}{T} \alpha bp \right) \right]$$
$$\cdot \left[-\frac{\alpha p q^2 T}{t_m{}^3} f\left(\frac{T}{t_m} q - (a - bp) \right) - \left(\frac{\alpha bp}{T} \right)^2 (1 - \alpha) p f\left(q - (a - bp) - \frac{T - t_m}{T} \alpha bp \right) \right]$$
$$- \left(\frac{\alpha p q T}{t_m{}^2} f\left(\frac{T}{t_m} q - (a - bp) \right) - \frac{1}{T} f\left(q - (a - bp) - \frac{T - t_m}{T} \alpha bp \right) \right)^2.$$

$\mathbb{D}$ can be summarized as follows:

$$\mathbb{D} = \mathcal{A} \cdot f\left(\frac{T}{t_m} q - (a - bp) \right)^2 + \mathcal{B} \cdot f\left(\frac{T}{t_m} q - (a - bp) \right) f\left(q - (a - bp) - \frac{T - t_m}{T} \alpha bp \right)$$
$$+ \mathcal{C} \cdot f\left(q - (a - bp) - \frac{T - t_m}{T} \alpha bp \right)^2.$$

Coefficient $\mathcal{A}$ is canceled to 0. In the case of $\mathcal{B}$, all coefficients have nonnegative conditions. Coefficient $\mathcal{C}$ consists of $(((1 - \alpha) \cdot \alpha)^2 p^2 (\frac{bp}{T})^2 - \frac{1}{T^2})$. The first term is always nonnegative. The second term can be offset by part of the coefficient of $\mathcal{B}$, $\frac{2\alpha pq}{t_m{}^2} f(\frac{T}{t_m} q - (a - bp)) f(q - (a - bp) - \frac{T - t_m}{T} \alpha bp)$. Since inequalities $pq \gg \alpha$, $T \geq t_m$, and $2f(\frac{T}{t_m} q - (a - bp)) \geq f(q - (a - bp) - \frac{T - t_m}{T} \alpha bp)$ hold true, coefficient $\mathcal{C}$ is nonnegative. Accordingly, the Hessian matrix $\mathbb{H}$ is negative definite and the profit function has the unique combination (t_m^{**}, q^{**}), which is the maximizer of the function.

Appendix E

As shown in Observation 3, the manufacturer's maximum profit occurs when the retailer starts the markdown sale at $t_m = 0$. Consider the condition that the expected profit function of the retailer decreases with increasing t_m that the first derivative of Π_r by t_m is negative.

$$\frac{\partial \Pi_r(q, t_m)}{\partial t_m} = \frac{\alpha p}{T} \left[(a - bp) F\left(\frac{T}{t_m} q - (a - bp) \right) - (1 - \alpha) bp F\left(q - (a - bp) - \frac{T - t_m}{T} \alpha bp \right) \right.$$
$$\left. + \int_{-\infty}^{\frac{T}{t_m} q - (a - bp)} x f(x) dx \right].$$

Recall Equation (3),

$$\alpha F\left(\frac{T}{t_m}q^* - (a - bp)\right) + (1-\alpha)F\left(q^* - (a - bp) - \frac{T - t_m}{T}\alpha bp\right) = \frac{p - (c_r + w)}{p}.$$

When the retailer places an optimal order quantity q^* from her standpoint, the above equation holds. By substituting the second term of the left-hand side with the first derivative of the profit function by t_m, it can be represented as follows:

$$\frac{\alpha p}{T}\left((a - bp + \alpha bp)F\left(\frac{T}{t_m}q^* - (a - bp)\right) - b(p - c_r - w) + \int_{-\infty}^{\frac{T}{t_m}q^* - (a-bp)} xf(x)dx\right).$$

Let the first derivative of the expected profit function with respect to t_m becomes negative.

$$(a - bp + \alpha bp)F\left(\frac{T}{t_m}q^* - (a - bp)\right) - b(p - c_r - w) + \int_{-\infty}^{\frac{T}{t_m}q^* - (a-bp)} xf(x)dx < 0.$$

The above inequality holds true even when the maximum value of the left-hand side has a negative value. To show the maximum value of the left-hand side, let q^* as the upper bound and t_m as the limit to 0:

$$a - bp + \alpha bp - bp + bc_r + bw < 0.$$

For the wholesale price w, a sufficient condition in Corollary 2 can be shown:

$$w < 2p - c_r - \alpha p - \frac{a}{b}$$

Appendix F

Let the first derivative of the expected profit function with respect to t_m become positive when t_m equals T:

$$(a - bp + \alpha bp)F\left(\frac{T}{t_m}q^* - (a - bp)\right) - b(p - c_r - w) + \int_{-\infty}^{q^* - (a-bp)} xf(x)dx > 0.$$

When t_m equals T, q^* equals $F^{-1}(\frac{p-(c_r+w)}{p}) + a - bp$ by Corollary 1. Thus, the following inequality holds true, which is the sufficient condition shown in Corollary 3:

$$(a - bp + \alpha bp)\left(\frac{p - (c_r + w)}{p}\right) - b(p - c_r - w) + \int_{-\infty}^{F^{-1}\left(\frac{p-(c_r+w)}{p}\right)} xf(x)dx > 0.$$