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A mobile multi-agent sensing problem with submodular functions under a partition matroid

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ABSTRACT

Multi-agent systems are generally applicable in a wide diversity of domains, such as robot engineering, computer science, the military, and smart cities. In this paper, we introduce a mobile multi-agent sensing problem and present a mathematical formulation. The model can be represented as a submodular maximization problem under a partition matroid constraint, which is NP-hard in general. The optimal solution of the model can be considered computationally intractable. Therefore, we propose two decent algorithms based on the greedy approach, which are global greedy and sequential greedy algorithms, respectively. We show that the sequential greedy algorithm is competitive with the global greedy algorithm and has advantages of computation times. Moreover, we present new approximation ratios of the sequential greedy algorithm and prove tightness of the ratios. Finally, we demonstrate the performances of our results through numerical experiments.

1. Introduction

One of the purposes for exploiting multi-agent systems is to monitor a set of nodes to detect event occurrences with a set of agents. The system can be applied in diverse domains such as wireless sensor networks in the military (Durišić et al., 2012), microgrid control in the energy field (Kantamneni et al., 2015), artificial intelligence (AI) in computer science (Stone and Veloso, 2000), medical asset tracking in hospital environments (Pietrabissa et al., 2013), and target tracking in robot engineering (Spletzer and Taylor, 2003). This problem can be expanded to radiation surveillance using unmanned aerial vehicles (UAVs) (Pöllönen et al., 2009), and to path planning in emergency management (Berger, 2015) as well. These multi-agent systems have been recently applied in smart cities, especially in environment monitoring systems (Jamil et al., 2015), urban-traffic management systems (Zhu et al., 2016), and in the improvement of servicing the internet of things (Verma et al., 2014).

When it comes to research on how multi-agent system can be applied, researchers have recently started to focus on mobile and heterogeneous agents. If an agent as a sensor moves around, the agent might cover initially uncovered locations at a later time, and the targets that might not be detected using stationary sensors can be detected (Liu et al., 2012). When we exploit mobile sensors, we can compensate for the lack of sensors and improve network coverage (Liu et al., 2005). For example, in the barrier coverage problem, a cost-effective system can be designed, in practice, by using mobile sensors (He et al., 2012). In addition, the city uses a group of UAVs as mobile agents that are equipped with an air quality measurement system instead of with stationary sensors. In particular, due to the emerging sensing technologies of IoT and the commercialization of drones, more and more fields have recently begun to utilize mobile sensors in the sensing problem. Therefore, we consider that a small number of agents can cover a wide range of areas by continuously moving around.

Using heterogeneous agents in multi-agent systems means using different types of agents. Nowadays, more and more studies related to the heterogeneous agents have been conducted in a wireless sensor networks (WSN) (Carrabs, 2015; Soeanu et al., 2018). In fact, implementing heterogeneous multi-agent systems offers several advantages, including higher versatility, cost reduction, and flexibility (Song et al., 2014). Heterogeneous agents also are shown to have better performance than homogeneous agents, because they take advantage of the strengths of each configuration (Scheutz et al., 2005). Although many studies have been devoted to multi-agent systems, little attention has been paid to heterogeneous mobile multi-agent systems. Therefore, we assume that

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our system model consists of heterogeneous agents with different sensing ranges and allowable distances to reflect general circumstances.

There are two types of sensing models: the deterministic sensing model (Boolean sensing) and the probabilistic sensing model (the Elfes sensing models) in the sensor planning problem (Hossain et al., 2012; Elfes, 1989; Karatas and Onggo, 2019). In this paper, we consider the problem with the Elfes sensing model because the effective sensing radius of an agent is affected by sensing device characteristics and environmental factors, which leads to non-uniform sensing (Rai and Daruwala, 2016). By considering Elfes sensing model, we can calculate the sum of event detection probabilities from all nodes. Each agent can move within its allowable distance in the next period to detect the events from the nodes. We call this problem a mobile multi-agent sensing problem. The objective function of the problem, as the sum of the detection probabilities, is monotone increasing and submodular. Therefore, the problem is represented as a submodular maximization problem under a partition matroid constraint, which is NP-hard in general. This problem has been solved through the greedy algorithm. The algorithm is known to achieve a $\frac{1}{2}$-approximation for the problem (Fisher et al., 1978; Kapralov et al., 2013).

In this paper, we deal with a mobile multi-agent sensing problem, which corresponds to a submodular maximization problem under a partition matroid constraint. We propose two decent algorithms based on the greedy approach (Fisher et al., 1978; Qu et al., 2019). The two algorithms are global greedy and sequential greedy algorithms, respectively. The global greedy algorithm corresponds to the general greedy algorithm. The sequential greedy algorithm, which is a variant of the global greedy algorithm, fixes the order of agents' arrivals before solving the problem and selects a strategy in the order of agents. This variant is similar in approach to the online setting in which the information of the agents is revealed sequentially and, on arrival of each agent's information, a strategy is chosen irrevocably. We show that the sequential greedy algorithm is competitive with the global one and has advantages of cost savings caused by time consumption. Another contribution of the paper is to present new approximation ratios of the sequential greedy algorithm, which might give tighter upper bounds. Beyond a worst-case $\frac{1}{2}$-approximation ratio, instance dependent guarantees are introduced to show improved bounds by using the concept of the *curvature* of the submodular function (Conforti and Cornuéjols, 1984). In addition to introducing the concept of *curvature* ($\frac{1}{1+c_1}$), we show new approximation ratios of the sequential greedy algorithm ($\frac{1}{1+h}$, R). We prove tightness of the ratios by presenting instances that the approximation ratios are achieved. In this paper, we present the novelty and validity of the new approximation ratios of the sequential greedy algorithm compared to the existing approximation ratio through both theoretical and experimental approaches.

1.1. Related work

The mobile multi-agents are generally equipped with sensing, computing, and communication devices; they also interact with each other (Olfati-Saber et al., 2007). To verify the application of multiple agents in complex environments, simulation models have been used (Luke et al., 2005; Castella, 2005; Li et al., 2010; Douma, 2012). There has been considerable interest in the analysis of multiple agents from an optimization perspective. Isler and Bajcsy (2005) addressed a probabilistic approach to solve the problem of selecting sensors to minimize the error in estimating the position of a target. Fei (2007) selected sensor locations that maximize information gain. Nedic and Ozdaglar (2009) presented an analysis for optimizing the sum of the convex objective functions corresponding to multiple agents. The probabilistic sensing model problems have recently been used in consideration of the submodular property (Krause et al., 2008; Clark and Poovendran, 2011; Krause and Guestrin, 2011; Zivan et al., 2015; Sun et al., 2017; Corah and Michael, 2018; Rezazadeh and Kia, 2019).

The submodular maximization problem under a matroid constraint has historically been solved through greedy-type algorithms. In particular, approximation ratios that give lower bounds, compared to the optimal solution, are generally used to measure the performance of the algorithms. Some papers presented algorithms based on the greedy approach, which gives a $\frac{1}{2}$-approximation (Nemhauser et al., 1978; Fisher et al., 1978; Lehmann et al., 2006; Rezazadeh and Kia, 2019), while the algorithm is known to be $\left(1-\frac{1}{e}\right)$-approximation for special cases (e.g., the uniform matroid).

Randomized algorithms and modified continuous greedy algorithms have been designed to give better approximation ratios in a theoretical way (Dobzinski and Schapira, 2006; Vondrák, 2008; Calinescu et al., 2011; Buchbinder et al., 2014; Sviridenko et al., 2017). In the online setting, Buchbinder et al. (2019) proved a 0.5096-competitiveness for the greedy algorithm in random order. In practice, however, the greedy algorithm is good enough to show much better performance than the existing approximation ratios. Thus, instance dependent guarantees have been introduced to show better performances, depending on the instances. Conforti and Cornuéjols (1984) presented the concept of *curvature*. The approximation ratio of the greedy algorithm is $\frac{1}{1+c_1}$ ($0 \leqslant c_1 \leqslant 1$), which is larger than $\frac{1}{2}$. After that, *element curvature*, *partial curvature*, and *discriminant* are designed as improved instance dependent guarantees (Wang et al., 2016; Sun et al., 2019; Liu et al., 2019; Rajaraman and Vaze, 2018).

There has recently been literature on the modified greedy algorithms to lessen computational complexity. As the size of agents increases, the computation time can be exponentially larger, even in greedy algorithms. Gharesifard and Smith (2017) and Rajaraman and Vaze (2018) presented a sequential distributed greedy algorithm in which the agents take their decision sequentially. This algorithm can be applied even in an online setting. Qu et al. (2019) compared the global greedy algorithm with the distributed greedy algorithm in terms of performance and computation time. The distributed greedy algorithm is a distributed variant of the global greedy algorithm, which adds local communication. The instance dependent guarantees can also be applied in these modified algorithms.

The mobile multi-agent sensing problem is an important problem in terms of operations management. In this paper, we conducted theoretical and experimental analysis of this problem. In comparison to previous studies, we design a sequential greedy algorithm to solve our problem, in which time complexity to obtain a greedy solution is less than that of the global greedy algorithm. We also prove new approximation ratios of the algorithm and show that the bound is tight even in the sequential greedy algorithm. In addition to theoretical contributions, we present the validity of using the new approximation ratios of the sequential greedy algorithm compared to the existing approximation ratio through numerical experiments.

The remainder of the paper is organized as follows. Section 2 presents the mobile multi-agent sensing problem mathematically and shows that the problem is a submodular maximization problem under a partition matroid constraint. In Section 3, the global and sequential greedy algorithms are presented to solve the problem. We also prove the new approximation ratios of the algorithms and their tightness. Section 4 provides numerical results of the two algorithms and shows the validity of the sequential greedy algorithm in terms of solution quality and computation times. Section 5 describes the contributions and conclusions of the paper.

2. Problem statement

In this section, we define the notations and describe a mobile multi-agent sensing problem mathematically. There are a set of $A = \{1, 2, \ldots, M\}$ mobile heterogeneous agents and a set of $B = \{1, 2, \ldots, N\}$ nodes. We set $i \in A$ and $j \in B$ to denote an agent and a node, respectively. Agents are deployed to monitor a set of nodes on a given space $\Omega \subset \mathbb{R}^2$. We assume $N \gg M$. The location of node j is o_j and the current location of agent i is l_i^c ($o_j, l_i^c \in \Omega$). In the next period, each agent can move to detect event occurrences from the nodes. The maximum distance that agent i can move during a unit period is defined as AL_i. Thus, $\mathscr{X}_i = \{(i,l_i) | \|l_i - l_i^c\| \leqslant AL_i\}$ is the set of the strategies for agent i and let $\mathscr{X} = \mathscr{X}_1 \cup \mathscr{X}_2 \cup \cdots \cup \mathscr{X}_M$. The probability of event occurrences at node j is E_j. Each agent i has its own bounded sensing radius δ_i. We assume that the sensing technique follows the Elfes sensing model (Elfes, 1989). Under a strategy $x = (i, l_i)$, the probability that agent i detects an event occurrence at node j is defined as

$$p\left(x,j\right) = \begin{cases} \exp(-\lambda_i \|l_i - o_j\|), & \text{if } \|l_i - o_j\| \leqslant \delta_i \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$\begin{aligned} f\left(\overline{\mathscr{X}}_1 \cup \left\{x\right\}\right) &= \sum_{j\in B} E_j \times \left(1 - \prod_{k\in \overline{\mathscr{X}}_1 \cup \{x\}} (1 - p(k,j))\right) \\ &= \sum_{j\in B} E_j \times \left(1 - (1-p(x,j))\prod_{k\in \overline{\mathscr{X}}_1} (1 - p(k,j))\right) \\ &= \sum_{j\in B} E_j \times \left(1 - \prod_{k\in \overline{\mathscr{X}}_1} (1 - p(k,j))\right) + \sum_{j\in B} E_j \times p\left(x,j\right)\prod_{k\in \overline{\mathscr{X}}_1} (1 - p(k,j)) \\ &= f\left(\overline{\mathscr{X}}_1\right) + \sum_{j\in B} E_j \times p\left(x,j\right)\prod_{k\in \overline{\mathscr{X}}_1} (1 - p(k,j)) \end{aligned}$$

where λ_i is a sensing decay factor of agent i. The characteristics of each agent are determined by AL_i, δ_i, and λ_i. Then, when $\overline{\mathscr{X}} \subseteq \mathscr{X}$, the joint probability that an event at node j is detected by a strategy set $\overline{\mathscr{X}}$ is calculated by

$$P_j\left(\overline{\mathscr{X}}\right) = 1 - \prod_{x\in\overline{\mathscr{X}}} \left(1 - p\left(x,j\right)\right) \tag{2}$$

where it is assumed that the detection probability of each agent is independent. The sum of event detection probabilities from all nodes can be represented as

$$\sum_{j\in B} E_j \times P_j\left(\overline{\mathscr{X}}\right) \tag{3}$$

We define $E_j \times P_j(\overline{\mathscr{X}})$ as a set function $f_j(\overline{\mathscr{X}})$, which means $f_j : 2^{\mathscr{X}} \to \mathbb{R}$. The set function f_j is normalized ($f_j(\varnothing) = 0$).

The objective of the problem is to find a strategy set of all agents, such that the Eq. (3) is maximized. We formulate the mobile multi-agent sensing problem as follows:

$$\max_{\overline{\mathscr{X}}} f\left(\overline{\mathscr{X}}\right) = \sum_{j\in B} f_j\left(\overline{\mathscr{X}}\right) \tag{4}$$

$$\begin{aligned} &\text{subject to } |\overline{\mathscr{X}} \cap \mathscr{X}_i| \leqslant 1, \forall i \in A \\ &\left(\text{subject to } \overline{\mathscr{X}} \in \mathscr{I}, \mathscr{I} = \left\{\mathscr{S} | \mathscr{S} \subseteq \mathscr{X} \text{ and } |\mathscr{S} \cap \mathscr{X}_i| \leqslant 1, \forall i \in A\right\}\right) \end{aligned} \tag{5}$$

where $\mathscr{I}$ is a non-empty collection of subsets of the set $\mathscr{X}$. An ordered pair $\mathscr{M} = (\mathscr{X}, \mathscr{I})$, where $\mathscr{I} \subseteq 2^{\mathscr{X}}$, is called a matroid if (a) for all $D \in \mathscr{I}$, any set $C \subseteq D$ is also in $\mathscr{I}$ and (b) for any $C, D \in \mathscr{I}$ and $|C| < |D|$, there exists a $j \in D \backslash C$ such that $C \cup \{j\} \in \mathscr{I}$. In this system, Constraint (5) is called a *partition matroid* (Nemhauser et al., 1978. The feasibility condition is to choose a strategy set that includes at most one strategy from each disjoint set $\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_M$. Theorem 1 shows that the objective function in this problem is a monotone increasing and submodular set function.

Definition 1. Given a ground set X, a set function $f : 2^X \to \mathbb{R}$ is defined to be monotone (increasing) if for any $S \subset T \subseteq X$, $f(S) \leqslant f(T)$, and submodular if for any $S \subset T \subseteq X$ and $x \notin T$, $f(T \cup \{x\}) - f(T) \leqslant f(S \cup \{x\}) - f(S)$.

Theorem 1. *The objective function (4) is monotone and submodular.*

Proof. Let $\overline{\mathscr{X}}_1$ and $\overline{\mathscr{X}}_2$, such that $\overline{\mathscr{X}}_1 \subseteq \overline{\mathscr{X}}_2 \subseteq \mathscr{X}$, be two strategy sets. Because $f_j(\overline{\mathscr{X}}_1) \leqslant f_j(\overline{\mathscr{X}}_2)$ $\forall j \in B$, we have $f(\overline{\mathscr{X}}_1) \leqslant f(\overline{\mathscr{X}}_2)$. It means that the function f is monotone. For $x \in \mathscr{X}$ and $x \notin \overline{\mathscr{X}}_2(\overline{\mathscr{X}}_1)$,

Therefore, $f(\overline{\mathscr{X}}_1 \cup \{x\}) - f(\overline{\mathscr{X}}_1) = \sum_{j\in B} E_j \times p(x,j) \prod_{k\in\overline{\mathscr{X}}_1} (1-p(k,j))$. Likewise, we can calculate $f(\overline{\mathscr{X}}_2 \cup \{x\}) - f(\overline{\mathscr{X}}_2) = \sum_{j\in B} E_j \times p(x,j) \prod_{k\in\overline{\mathscr{X}}_2} (1-p(k,j))$. Because $\prod_{k\in\overline{\mathscr{X}}_1} (1-p(k,j)) \geqslant \prod_{k\in\overline{\mathscr{X}}_2} (1-p(k,j))$ $\forall j \in B$, $f(\overline{\mathscr{X}}_1 \cup \{x\}) - f(\overline{\mathscr{X}}_1) \geqslant f(\overline{\mathscr{X}}_2 \cup \{x\}) - f(\overline{\mathscr{X}}_2)$. Therefore, the function f is submodular. □The problem can be represented as a submodular maximization problem under a partition matroid constraint. Maximizing a submodular function under a matriod constraint is a member of the class of NP-hard problems (Nemhauser et al., 1978; Fisher et al., 1978). Even for special cases such as uniform matroid and partition matroid, the submodular maximization problem is known to be NP-hard (Rajaraman and Vaze, 2018; Lovász, 1983).

3. Algorithms and approximation ratios

We know that the mobile multi-agent sensing problem is a submodular maximization problem under a partition matroid constraint. The feasible region of the problem is exponentially large in the size of M and N. In this case, the optimal solution can be intractable to compute within a reasonable time. The greedy algorithm was implemented to solve the problem in the previous research. For general cases, the algorithm is known to give an approximation ratio of $1/2$, which means the objective value the algorithm presents is at least $1/2$ of the optimal objective value (Fisher et al., 1978). Under a uniform matroid constraint, some papers presented an improved approximation ratio of $\left(1 - \frac{1}{e}\right)$ using the greedy-based algorithms (Nemhauser et al., 1978; Fisher et al., 1978; Lehmann et al., 2006; Rezazadeh and Kia, 2019). However, there has been little research on giving an improved approximation ratio in the mobile multi-agent sensing problem under a partition matroid constraint. This paper presents two types of greedy algorithms to give improved approximation ratios.

3.1. Global greedy algorithm

Algorithm 1 shows the procedure of the global greedy algorithm, based on the general greedy approach.

Algorithm 1. Global greedy algorithm

Algorithm 1: Global greedy algorithm

Input : $\mathcal{X}$ ($= \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_M$, $\mathcal{X}_i$: set of the strategies for agent i)
Output: $\overline{\mathcal{X}}$
$\overline{\mathcal{X}} \leftarrow \emptyset$, $t \leftarrow 1$;
while $t \leq M$ **do**
  $k^* = \mathrm{argmax}_{k \in \mathcal{X}}(f(\overline{\mathcal{X}} \cup \{k\}) - f(\overline{\mathcal{X}}))$;
  (A strategy is arbitrary selected if two or more strategies have the same value.)
  $\overline{\mathcal{X}} \leftarrow \overline{\mathcal{X}} \cup \{k^*\}$;
  $p \leftarrow$ the first element of k^* such that $k^* = (p, l_p)$;
  $\mathcal{X} \leftarrow \mathcal{X} \setminus \mathcal{X}_p$;
  $t \leftarrow t + 1$;
end

In each step, a strategy that provides the largest marginal gain from the current state is added while satisfying Constraint (5). The number of strategies is infinite, because the feasible region of the problem includes infinite points. To compute the problem within finite iterations, we restrict the infinite strategies to finite points. The algorithm calculates at most $|\mathscr{X}|$ times in each step. We call this algorithm a global greedy algorithm because all possible strategies have to be considered in each step. The time complexity of the global greedy algorithm is $\mathbf{O}(M|\mathscr{X}|)$.

3.2. Sequential greedy algorithm

We present a decent algorithm based on the sequential greedy algorithm (Fisher et al., 1978; Rezazadeh and Kia, 2019). A sequence of a set of agents A, which is a permutation over M agents, has to be decided before executing the algorithm. There are various methods to decide the sequence of the set, such as a random method, a method based on the current location of the agents, and a method based on the marginal gain of each agent. In this paper, we adopt the random method to investigate the effectiveness of the sequential greedy algorithm itself. When the other methods are applied, initial solutions, rather than the sequential greedy algorithm, may affect the quality of objective value. We assume that the sequence of the agents is $(1, 2, \ldots, M)$. The proposed algorithm is referred to as a sequential greedy algorithm, which is shown in Algorithm 2. The algorithm can correspond to the greedy algorithm for an online version of the problem, which means that the algorithm decides a strategy of the current agent without knowing the strategies of the agents not yet considered. In the t^{th} step, a strategy of agent t that provides the largest marginal gain from the current state is added while satisfying Constraint (5). This means that the algorithm determines a strategy for agent t without considering the information about agents $t+1, \ldots, M$. The time complexity of the sequential greedy algorithm is $\mathbf{O}(MH)$, where $H = \max \mathscr{X}_i \; \forall i \in A$. Because $M|\mathscr{X}| \geqslant MH$, Algorithm 2 is faster than Algorithm 1. The time difference between the two algorithms becomes larger as M and N become larger.

Algorithm 2. Sequential greedy algorithm

Algorithm 2: Sequential greedy algorithm

Input : $\mathcal{X}$ ($= \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_M$, $\mathcal{X}_i$: set of the strategies for agent i)
  a set of agents A associated with an ordering $(1, 2, \ldots, M)$
Output: $\overline{\mathcal{X}}$
$\overline{\mathcal{X}} \leftarrow \emptyset$, $t \leftarrow 1$;
while $t \leq M$ **do**
  $k^* = \mathrm{argmax}_{k \in \mathcal{X}_t}(f(\overline{\mathcal{X}} \cup \{k\}) - f(\overline{\mathcal{X}}))$;
  (A strategy is arbitrary selected if two or more strategies have the same value.)
  $\overline{\mathcal{X}} \leftarrow \overline{\mathcal{X}} \cup \{k^*\}$;
  $t \leftarrow t + 1$;
end

Fig. 1 is a flow chart showing the procedure of the two greedy algorithms. The input parameters include the location and event occurrence probability of the nodes. The location, allowable distance, and sensing decay factor of the agents are also included in the input parameters. The global greedy algorithm proceeds with search, insertion, deletion, and increase in every iteration. On the other hand, the sequential greedy algorithm determines the sequence of agents, then iterates through the search, insertion, and increase processes.

3.3. Approximation ratio

The sequential greedy algorithm is also known to give an approximation ratio of $1/2$ (Fisher et al., 1978), which is the same ratio of the global greedy algorithm. The ratio $1/2$ is a worst-case bound, but, in practice, the ratios of the greedy algorithm solution to an optimal solution are more likely to be close to 1 rather than $1/2$. Thus, instance dependent guarantees, like the concept of *curvature* or *discriminant*, have emerged to show improved bounds, depending on the instances (Conforti and Cornuéjols, 1984; Rajaraman and Vaze, 2018).

In this paper, we present and prove new instance dependent guarantees for the sequential greedy algorithm. Before introducing the instance dependent guarantees, we define some notations that will be used in the proof of the following theorems. Let $\overline{\mathscr{X}}^*$ and $\overline{\mathscr{X}}^G$ be the optimal strategy set of the problem and the strategy set generated by the sequential greedy algorithm. We also let O_i and G_i be the strategy of agent i in the optimal and greedy solution, respectively ($O_i := \overline{\mathscr{X}}^* \cap \mathscr{X}_i$ and $G_i := \overline{\mathscr{X}}^G \cap \mathscr{X}_i$). Let $\overline{\mathscr{X}}^*_S$ and $\overline{\mathscr{X}}^G_S$ be the union of the optimal and greedy strategy of agent 1, agent 2, …, agent S. ($\overline{\mathscr{X}}^*_S := \bigcup_{i=1}^S O_i$ and $\overline{\mathscr{X}}^G_S := \bigcup_{i=1}^S G_i$). We define $\rho_k(\overline{\mathscr{X}})$ as $f(\overline{\mathscr{X}} \cup \{k\}) - f(\overline{\mathscr{X}})$. The following theorems show new instance dependent guarantees and prove the ratios and tightness of them. When there exists a scenario such that the approximation ratio is achieved, we can prove tightness of the bound. The approximation ratios presented can also be applied in the global greedy algorithm.

$$f\left(\overline{\mathscr{X}}^*\right) \leqslant f\left(\overline{\mathscr{X}}^G\right) + \frac{1}{1-c_1}\sum_{k\in\overline{\mathscr{X}}^*\backslash\overline{\mathscr{X}}^G}\rho_k\left(\overline{\mathscr{X}}^G\right) - \sum_{k\in\overline{\mathscr{X}}^G\backslash\overline{\mathscr{X}}^*}\rho_k\left(\overline{\mathscr{X}}^G\backslash\left\{k\right\}\right)$$
$$= f\left(\overline{\mathscr{X}}^G\right) + \frac{c_1}{1-c_1}\sum_{k\in\overline{\mathscr{X}}^*\backslash\overline{\mathscr{X}}^G}\rho_k\left(\overline{\mathscr{X}}^G\right) + \sum_{k\in\overline{\mathscr{X}}^*\backslash\overline{\mathscr{X}}^G}\rho_k\left(\overline{\mathscr{X}}^G\right) - \sum_{k\in\overline{\mathscr{X}}^G\backslash\overline{\mathscr{X}}^*}\rho_k\left(\overline{\mathscr{X}}^G\backslash\left\{k\right\}\right)$$
$$\leqslant f\left(\overline{\mathscr{X}}^G\right) + \frac{c_1}{1-c_1}\sum_{k\in\overline{\mathscr{X}}^*\backslash\overline{\mathscr{X}}^G}\rho_k\left(\overline{\mathscr{X}}^G\right) \tag{10}$$

Theorem 2. *For the mobile multi-agent sensing problem, the approximation ratio of Algorithm 2 is* $\frac{f(\overline{\mathscr{X}}^G)}{f(\overline{\mathscr{X}}^*)} \geqslant \frac{1}{1+h}$ $\left(h := \frac{c_1\cdot c_2}{1-c_1}\right.$ *such that* $c_1 = \max_{k\notin\overline{\mathscr{X}},\overline{\mathscr{X}}\in\mathscr{I},j\in B}\left(1 - \frac{f_j(\overline{\mathscr{X}}\cup\{k\})-f_j(\overline{\mathscr{X}})}{f_j(\{k\})}\right)$ *and* $\left. c_2 = \max\frac{\sum_{t=1}^M \rho_{k_t}(\overline{\mathscr{X}}^G)}{f(\overline{\mathscr{X}}^G)}, k_t \in \mathscr{X}_t\right)$. *The ratio is acceptable when* $c_1 \neq 1$ *and* $c_1\cdot c_2 \leqslant 1 - c_1$ ($h \leqslant 1$). *The bound is also tight.*

Proof. We first present Lemma 1 to prove the ratio.

Lemma 1. *For any* $\overline{\mathscr{X}}_1 \subseteq \overline{\mathscr{X}}_2, \rho_k(\overline{\mathscr{X}}_2) \geqslant (1-c_1)\cdot\rho_k(\overline{\mathscr{X}}_1)$.

Proof.

$$\begin{aligned}\rho_k\left(\overline{\mathscr{X}}_2\right) &\geqslant (1-c_1)\cdot f(\{k\}) \because \text{definition of } c_1 \\ &= (1-c_1)\cdot\rho_k(\varnothing) \\ &\geqslant (1-c_1)\cdot\rho_k\left(\overline{\mathscr{X}}_1\right) \because \text{submodular} \quad \square\end{aligned}$$

We use $f(\overline{\mathscr{X}}^* \cap \overline{\mathscr{X}}^G)$ to prove the ratio.

$$\begin{aligned}f\left(\overline{\mathscr{X}}^* \cap \overline{\mathscr{X}}^G\right) &= f\left(\overline{\mathscr{X}}^G\right) - \sum_{t=1}^M \sum_{k\in G_t\backslash O_t}\rho_k\left(\left(\overline{\mathscr{X}}^* \cap \overline{\mathscr{X}}^G\right) \cup \overline{\mathscr{X}}^G_{t-1}\right) \\ &\leqslant f\left(\overline{\mathscr{X}}^G\right) - \sum_{t=1}^M \sum_{k\in G_t\backslash O_t}\rho_k\left(\overline{\mathscr{X}}^G\backslash\left\{k\right\}\right)\end{aligned} \tag{6}$$

$$= f\left(\overline{\mathscr{X}}^G\right) - \sum_{k\in\overline{\mathscr{X}}^G\backslash\overline{\mathscr{X}}^*}\rho_k\left(\overline{\mathscr{X}}^G\backslash\left\{k\right\}\right) \tag{7}$$

Inequality (6) follows from the submodularity of f.

$$\begin{aligned}f\left(\overline{\mathscr{X}}^* \cap \overline{\mathscr{X}}^G\right) &= f\left(\overline{\mathscr{X}}^*\right) - \sum_{t=1}^M \sum_{k\in O_t\backslash G_t}\rho_k\left(\left(\overline{\mathscr{X}}^* \cap \overline{\mathscr{X}}^G\right) \cup \overline{\mathscr{X}}^*_{t-1}\right) \\ &\geqslant f\left(\overline{\mathscr{X}}^*\right) - \sum_{t=1}^M \sum_{k\in O_t\backslash G_t}\rho_k\left(\left(\overline{\mathscr{X}}^* \cap \overline{\mathscr{X}}^G\right)\right)\end{aligned} \tag{8}$$

$$\geqslant f\left(\overline{\mathscr{X}}^*\right) - \frac{1}{1-c_1}\sum_{k\in\overline{\mathscr{X}}^*\backslash\overline{\mathscr{X}}^G}\rho_k\left(\overline{\mathscr{X}}^G\right) \tag{9}$$

Inequality (8) follows from the submodularity of f. Inequality (9) is due to Lemma 1. Combining the two (In) equalities (7) and (9),

$$\leqslant f\left(\overline{\mathscr{X}}^G\right) + \frac{c_1\cdot c_2}{1-c_1}f\left(\overline{\mathscr{X}}^G\right) \tag{11}$$

$$= \left(1+h\right)f\left(\overline{\mathscr{X}}^G\right)$$

Because $\rho_{k'}(\overline{\mathscr{X}}^G) \leqslant \rho_k(\overline{\mathscr{X}}^G\backslash\{k\})$ and $\left|\overline{\mathscr{X}}^*\backslash\overline{\mathscr{X}}^G\right| = \left|\overline{\mathscr{X}}^G\backslash\overline{\mathscr{X}}^*\right|$, Inequality (10) is satisfied. Inequality (11) is due to the definition of c_2. Therefore, $\frac{f(\overline{\mathscr{X}}^G)}{f(\overline{\mathscr{X}}^*)} \geqslant \frac{1}{1+h}$. If $h > 1$, the approximation ratio becomes less than $\frac{1}{2}$. In this case, we use $\frac{1}{2}$ instead of $\frac{1}{1+h}$ as an approximation ratio.

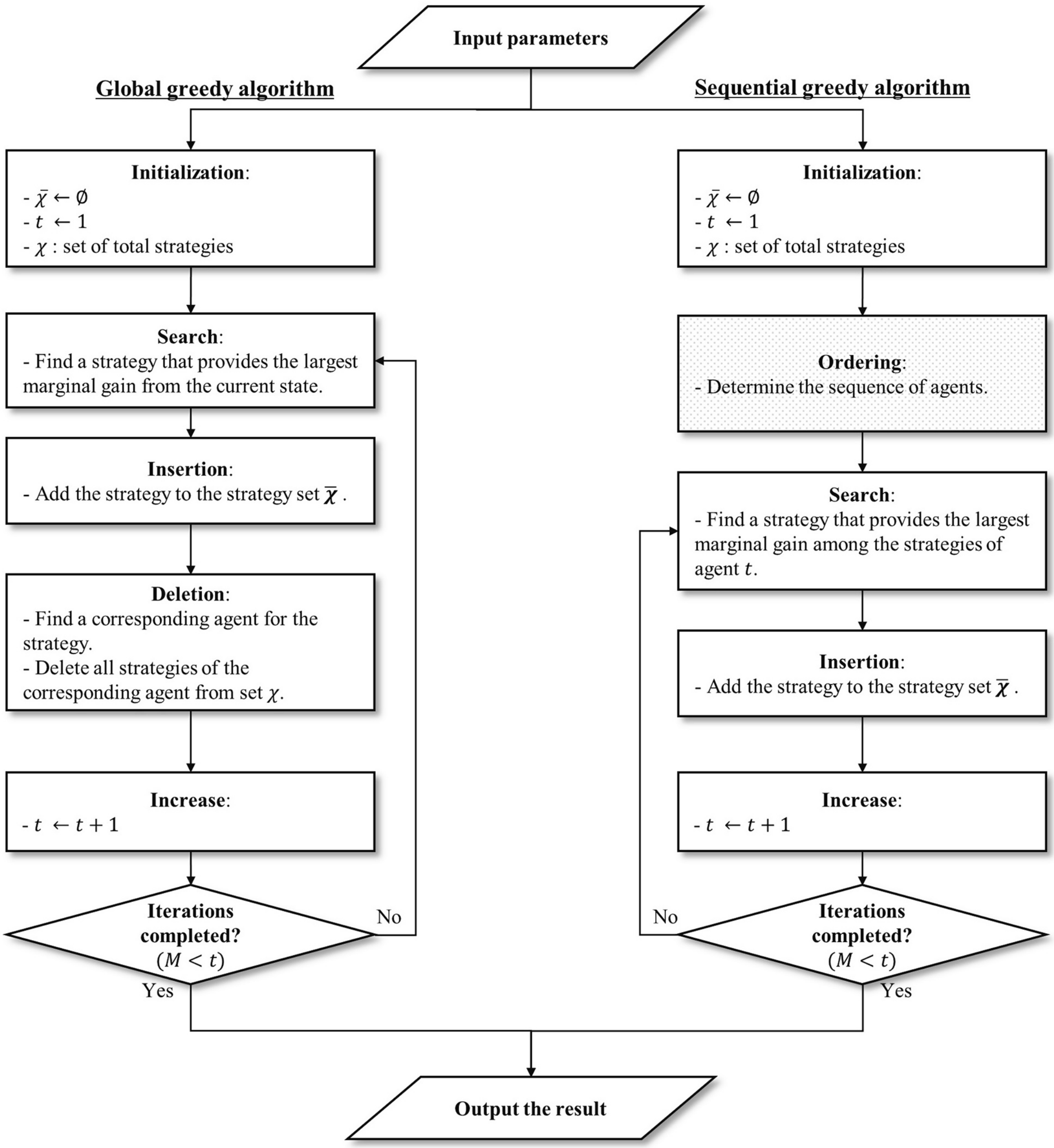

Fig. 1. Flow chart for the two greedy algorithms.

Next, we prove tightness of the bound in the sense that there exist instances that approximation ratio $\frac{1}{1+h}$ is achieved. When $c_1 = 0$, the submodular function f becomes linear. As a result, the solution given by Algorithm 2, $\overline{\mathscr{X}}^G$, is optimal, and the ratio is trivially 1. When $c_1 \neq 0$, all instances such that $c_2 = 0$ achieve tightness. In these instances, the ratio becomes 1. It means that the solution given by Algorithm 2, $\overline{\mathscr{X}}^G$, is optimal. We establish the claim by contradiction. Suppose that $\overline{\mathscr{X}}'$ is an optimal solution (not $\overline{\mathscr{X}}^G$) and $\left|\overline{\mathscr{X}}^G \backslash \overline{\mathscr{X}}'\right| = \left|\overline{\mathscr{X}}' \backslash \overline{\mathscr{X}}^G\right| = 1$ for the sake of simplicity. Let $\{a\} \in \overline{\mathscr{X}}^G \backslash \overline{\mathscr{X}}'$ and $\{b\} \in \overline{\mathscr{X}}' \backslash \overline{\mathscr{X}}^G$. Because we assume $c_2 = 0, f(\overline{\mathscr{X}}^G) = f(\overline{\mathscr{X}}^G \cup \{b\})$. But we know $f(\overline{\mathscr{X}}' \cup \{a\}) = f(\overline{\mathscr{X}}^G \cup \{b\})$ by the assumption. Thus $f(\overline{\mathscr{X}}' \cup \{a\}) = f(\overline{\mathscr{X}}^G)$. It means that $f(\overline{\mathscr{X}}' \cup \{a\}) = f(\overline{\mathscr{X}}^G) \leqslant f(\overline{\mathscr{X}}')$. The function f is monotone, so $f(\overline{\mathscr{X}}' \cup \{a\}) \geqslant f(\overline{\mathscr{X}}')$. Therefore, $f(\overline{\mathscr{X}}' \cup \{a\}) = f(\overline{\mathscr{X}}') = f(\overline{\mathscr{X}}^G)$, which is a contradiction. □

We present the following lemma by modifying Conforti and Cornuéjols (1984)'s theorem to apply our problem.

Lemma 2. (Conforti and Cornuéjols, 1984) *For the mobile multi-agent sensing problem, the approximation ratio of Algorithm 2 is* $\frac{f(\overline{\mathscr{X}}^G)}{f(\overline{\mathscr{X}}^*)} \geqslant \frac{1}{1+c_1}$.

Proof.

$$f\left(\overline{\mathscr{X}}^* \cup \overline{\mathscr{X}}^G\right) \leqslant f\left(\overline{\mathscr{X}}^G\right) + \sum\nolimits_{k \in \overline{\mathscr{X}}^* \backslash \overline{\mathscr{X}}^G} \rho_k\left(\overline{\mathscr{X}}^G\right) \tag{12}$$

$$\begin{aligned} &= f\left(\overline{\mathscr{X}}^G\right) + \sum_{t=1}^{M} \sum_{k \in O_t \backslash G_t} \rho_k\left(\overline{\mathscr{X}}^G\right) \\ &\leqslant f\left(\overline{\mathscr{X}}^G\right) + \sum_{t=1}^{M} \sum_{k \in O_t \backslash G_t} \rho_k\left(\overline{\mathscr{X}}^G_{t-1}\right) \end{aligned} \tag{13}$$

We have Inequality (12) from Lemma 2.1 of Conforti and Cornuéjols (1984). Inequality (13) follows from the submodularity of f.

$$f\left(\mathscr{X}^* \cup \mathscr{X}^G\right) = f\left(\mathscr{X}^*\right) + \sum_{t=1}^{M} \sum_{k \in G_t \setminus O_t} \rho_k\left(\mathscr{X}^* \cup \mathscr{X}^G_{t-1}\right) \tag{14}$$

Combining the two (In) equalities (13) and (14),

$$\begin{aligned} f\left(\mathscr{X}^*\right) &\leqslant f\left(\mathscr{X}^G\right) + \sum_{t=1}^{M} \sum_{k \in O_t \setminus G_t} \rho_k\left(\mathscr{X}^G_{t-1}\right) - \sum_{t=1}^{M} \sum_{k \in G_t \setminus O_t} \rho_k\left(\mathscr{X}^* \cup \mathscr{X}^G_{t-1}\right) \\ &\leqslant f\left(\mathscr{X}^G\right) + \sum_{t=1}^{M} \sum_{k \in O_t \setminus G_t} \rho_k\left(\mathscr{X}^G_{t-1}\right) - \left(1 - c_1\right) \sum_{t=1}^{M} \sum_{k \in G_t \setminus O_t} \rho_k\left(\mathscr{X}^G_{t-1}\right) \end{aligned} \tag{15}$$

$$\leqslant f\left(\mathscr{X}^G\right) + \sum_{t=1}^{M} \sum_{k \in G_t \setminus O_t} \rho_k\left(\mathscr{X}^G_{t-1}\right) - \left(1 - c_1\right) \sum_{t=1}^{M} \sum_{k \in G_t \setminus O_t} \rho_k\left(\mathscr{X}^G_{t-1}\right) \tag{16}$$

$$\begin{aligned} &= f\left(\mathscr{X}^G\right) + c_1 \sum_{t=1}^{M} \sum_{k \in G_t \setminus O_t} \rho_k\left(\mathscr{X}^G_{t-1}\right) \\ &\leqslant f\left(\mathscr{X}^G\right) + c_1 \sum_{t=1}^{M} \sum_{k \in G_t} \rho_k\left(\mathscr{X}^G_{t-1}\right) \end{aligned} \tag{17}$$

$$\begin{aligned} &= f\left(\mathscr{X}^G\right) + c_1 f\left(\mathscr{X}^G\right) \\ &= \left(1 + c_1\right) f\left(\mathscr{X}^G\right) \end{aligned}$$

Inequality (15) is due to Lemma 1, and Inequality (16) follows from the property of the greedy algorithm. Inequality (17) follows from the monotonicity of f. Therefore, $\frac{f(\mathscr{X}^G)}{f(\mathscr{X}^*)} \geqslant \frac{1}{1+c_1}$. The proof of tightness of the bound is shown in *Conforti & Cornuéjols* (1984) and *Qu et al.* (2019), so we skip the details of the proof. □

Lemma 3. *When $c_1 + c_2 \leqslant 1$, using $\frac{1}{1+h}$ is more advantageous to obtain a tighter upper bound than using $\frac{1}{1+c_1}$.*

Proof. When $\frac{1}{1+h} \geqslant \frac{1}{1+c_1}$, we can say that $\frac{1}{1+h}$ is more acceptable when estimating an upper bound of the optimal. It means that $h \leqslant c_1$ ($\frac{c_1 \cdot c_2}{1-c_1} \leqslant c_1$). Therefore, when $c_1 + c_2 \leqslant 1$, using $\frac{1}{1+h}$ is more advantageous to obtain a tighter upper bound than using $\frac{1}{1+c_1}$. □

Theorem 3. *For the mobile multi-agent sensing problem, the approximation ratio of Algorithm 2 is $\frac{f(\mathscr{X}^G)}{f(\mathscr{X}^*)} \geqslant R \left(R := \min \frac{\rho_{G_t}(\mathscr{X}^G_{t-1})}{\max_{k \in \mathscr{X}_t} f(\{k\})}\right)$. The ratio is acceptable when $R \geqslant \frac{1}{2}$. The bound is also tight.*

Proof. We prove the ratio, by induction on the number of agents M, when $R \geqslant \frac{1}{2}$. When $M = 1$, $\frac{f(\mathscr{X}^G)}{f(\mathscr{X}^*)} = \frac{f(G_1)}{f(G_1)} \geqslant R$. Suppose that it is true for $M = n$. Let $\mathscr{X}^G_{M=n}$ be a solution generated by the sequential greedy algorithm when $M = n$. Let $\mathscr{X}^*_{M=n}$ be an optimal solution when $M = n$. We focus on $M = n + 1$. First of all, the following equality and inequality are satisfied:

$$\begin{aligned} f\left(\mathscr{X}^G_{M=n+1}\right) &= f\left(\mathscr{X}^G_{M=n}\right) + \rho_{G_{n+1}}\left(\mathscr{X}^G_{M=n}\right) \\ f\left(\mathscr{X}^*_{M=n+1}\right) &\leqslant f\left(\mathscr{X}^*_{M=n}\right) + \max_{k \in \mathscr{X}_{n+1}} f\left(\left\{k\right\}\right) \end{aligned} \tag{18}$$

because

$$\begin{aligned} f\left(\mathscr{X}^*_{M=n+1}\right) &= f\left(\mathscr{X}^*_{M=n} \cup O_{n+1}\right) + f\left(\mathscr{X}^*_{M=n} \cap O_{n+1}\right) \because f\left(\mathscr{X}^*_{M=n} \cap O_{n+1}\right) = 0 \\ &\leqslant f\left(\mathscr{X}^*_{M=n}\right) + f\left(O_{n+1}\right) \because \text{submodular} \\ &\leqslant f\left(\mathscr{X}^*_{M=n}\right) + \max_{k \in \mathscr{X}_{n+1}} f\left(\left\{k\right\}\right). \end{aligned}$$

Using the above equality and inequality, we can prove approximation ratio R.

$$\frac{f\left(\mathscr{X}^G_{M=n+1}\right)}{f\left(\mathscr{X}^*_{M=n+1}\right)} \geqslant \frac{f\left(\mathscr{X}^G_{M=n}\right) + \rho_{G_{n+1}}\left(\mathscr{X}^G_{M=n}\right)}{f\left(\mathscr{X}^*_{M=n}\right) + \max_{k \in \mathscr{X}_{n+1}} f\left(\left\{k\right\}\right)} \tag{19}$$

$$\geqslant \frac{R \times f\left(\mathscr{X}^*_{M=n}\right) + \rho_{G_{n+1}}\left(\mathscr{X}^G_{M=n}\right)}{f\left(\mathscr{X}^*_{M=n}\right) + \max_{k \in \mathscr{X}_{n+1}} f\left(\left\{k\right\}\right)} \tag{20}$$

$$\geqslant \frac{R \times f\left(\mathscr{X}^*_{M=n}\right) + R \times \max_{k \in \mathscr{X}_{n+1}} f\left(\left\{k\right\}\right)}{f\left(\mathscr{X}^*_{M=n}\right) + \max_{k \in \mathscr{X}_{n+1}} f\left(\left\{k\right\}\right)} \tag{21}$$

$$= R$$

Inequality (19) is due to Inequality (18). Inequality (20) follows from $\frac{f(\mathscr{X}^G_{M=n})}{f(\mathscr{X}^*_{M=n})} \geqslant R$. Inequality (21) is due to $\frac{\rho_{G_{n+1}}(\mathscr{X}^G_{M=n})}{\max_{k \in \mathscr{X}_{n+1}} f(\{k\})} \geqslant R$. When $R < \frac{1}{2}$, we use $\frac{1}{2}$ instead of R as an approximation ratio.

Next, we prove tightness of the bound in the sense that there exist instances that approximation ratio R is achieved. Let R be $\frac{\alpha}{\beta}$. It means that $\frac{\alpha}{\beta} \geqslant \frac{1}{2}$. We assume that there are M agents ($i \in A$) and N nodes ($j \in B$). We set $M \geqslant K$ such that K is the smallest positive integer with $K\alpha \leqslant (K-1)\beta$. We need some notations: $\mathscr{X}_i = \{(i,j) | j \in B\}$ is the set of the strategies for agent i. $S_j(\mathscr{X}) = \{i | (i,j) \in \mathscr{X}\}$ is the set of agents that select node j. For node j, $\mathscr{X} = Z^j_1 \cup Z^j_2$ where $Z^j_1 = \{k | k \in \mathscr{X}, i \in S_j(\mathscr{X})$ when $k = (i,j)\}$ and $Z^j_2 = \mathscr{X} - Z^j_1$. We assume that $f_j(\mathscr{X}) = f_j(Z^j_1)$. Suppose that a set of agents A associated with an ordering $(1, 2, \ldots, M)$. The set values are as follows:

(i) if $\left|Z^j_1\right| = 1$ in $\mathscr{X}$,

$$f_j\left(\mathscr{X}\right) = \begin{cases} \beta, & \text{if} Z^j_1 = \left\{\left(j+1, j\right)\right\}, j = 1, 2, \ldots M-1 \\ \alpha, & \text{otherwise} \end{cases}$$

(ii) if $\left|Z^j_1\right| = 2$ in $\mathscr{X}$, $f_j(\mathscr{X}) = \beta$

(iii) if $\left|Z^j_1\right| \geqslant 3$ in $\mathscr{X}$, $f_j(\mathscr{X}) = f_j(\mathscr{X} \setminus \{k\})$, $k \in Z^j_1$

The set functions $f_j(\mathscr{X})$ are all monotone submodular functions. In this setting,

$\min \frac{\rho_{G_t}(\mathscr{X}^G_{t-1})}{\max_{k \in \mathscr{X}_t} f(\{k\})} = \frac{\alpha}{\beta}$ is satisfied. We assume that the smallest index of node j is selected if two or more strategies have the same value. So, $\mathscr{X}^G = \{(i,j) | i = j \text{ and } i = 1, 2, \ldots, M\}$. However, $\mathscr{X}^* = \{(i,j) | i = j + 1 \text{ and } j = 1, 2, \ldots, M-1\} \cup \{(1, M)\}$. Therefore, $\frac{\sum_{j \in B} f_j(\mathscr{X}^G)}{\sum_{j \in B} f_j(\mathscr{X}^*)} = \frac{M\alpha}{M\beta - \beta + \alpha} \approx \frac{\alpha}{\beta}$ when $M \to \infty$. □

The following lemma shows the overall approximation ratio of Algorithm 2 by using Theorems 2 and 3, and Lemma 2.

Lemma 4. *For the mobile multi-agent sensing problem, the approximation ratio of Algorithm 2 is bounded by $\frac{f(\mathscr{X}^G)}{f(\mathscr{X}^*)} \geqslant \max\left\{\frac{1}{1+c_1}, \frac{1}{1+h}, R\right\}$.*

Proof. We can prove this lemma by using Theorems 2 and 3, and Lemma 2. □

Table 1
Results for the different sensing decay factor λ_i.

λ_i	$r_{gg/opt}$	$r_{sg/opt}$	t_{opt} (sec)	t_{gg} (sec)	t_{sg} (sec)
0.1	0.992	0.991	367.399	0.056	0.025
0.2	0.963	0.958	284.844	0.049	0.020
0.3	0.994	0.991	193.681	0.033	0.017
0.4	0.993	0.985	246.191	0.029	0.015
0.5	0.997	0.988	311.668	0.029	0.015

4. Computational experiments

In this section, the global and sequential greedy algorithms presented were evaluated through numerical tests. All tests were run on a Python 3 with Intel Core CPU i5-3470 processor. We considered a large number of N nodes and a relatively small number of M agents, both of which are located as points in a two-dimensional space $\mathbb{R}^2$. We generated the locations of agents and nodes uniformly random. The strategy set ($\mathscr{X}_i$) of agent i can include all the points whose distances to l_i^c are within AL_i. Because the number of strategy sets is infinite, the points are restricted to integers.

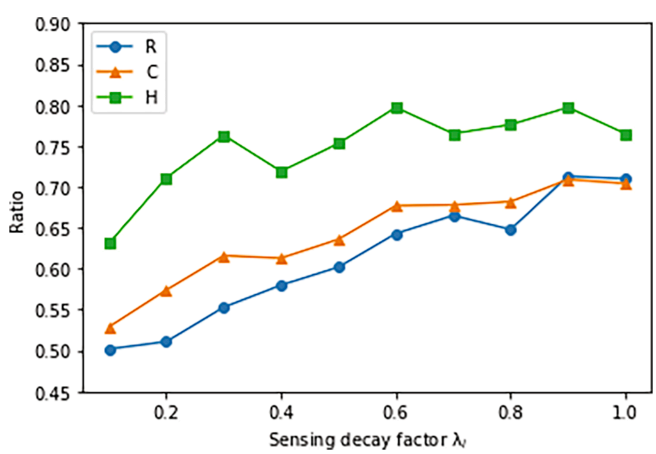

Fig. 2. Comparison between approximation ratios R, C, and H for λ_i.

4.1. Experiments in small data sets

First of all, we conducted numerical experiments to analyze the performance of the algorithms in small data sets. For small data sets, we chose $M = 5, N = 10$,and $AL_i \leqslant 3$. We set a sensing range of agent i according to AL_i. As AL_i is high, the sensing range is low. Agents and nodes are in $[0,20]^2$. We executed 100 runs for each sensing decay factor $\lambda_i \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. We computed the optimal solution through a brute-force search, and also derived solutions through the global greedy algorithm and the sequential greedy algorithm. For the sake of simplicity, we use $C := \frac{1}{1+c_1}$ and $H := \frac{1}{1+h}$ in this section.

Table 1 shows the results for the different sensing decay factors. The values in Table 1 are average ratios and computation times from the numerical experiments. $r_{gg/opt}$ and $r_{sg/opt}$ are ratios of the global and sequential greedy algorithm solutions to an optimal solution, respectively. There are three types of computation times: t_{opt} for the brute-force search method, t_{gg} for the global greedy algorithm, and t_{sg} for the sequential greedy algorithm. $r_{gg/opt}$ and $r_{sg/opt}$ showed larger than 0.98 in all cases. The ratio differences between the two algorithms were less than 1%. The computation times were less than 0.1 s in the algorithms, but more than 192 s in the brute-force search. The computation times of the sequential greedy algorithm were about half of the computation times of the global greedy algorithm.

Existing approximation ratio C was compared with new approximation ratios R and H, which were proved in this paper, through the numerical experiments. Fig. 2 is a plot of approximation ratios R, C, and H for the different sensing decay factor. We additionally executed 100 runs for each sensing decay factor $\lambda_i \in \{0.6, 0.7, 0.8, 0.9, 1.0\}$ to compare the approximation ratios. Approximation ratio H ranged between 0.631 and 0.763 in the numerical experiments. In most data sets, approximation ratio H was the largest ratio among the three approximation ratios. When λ_i was 0.9 and 1.0, approximation ratio R tended to be larger than approximation ratio C on average. We also observed that as λ_i increased, the three approximation ratios were more likely to increase. The sensing decaying factor λ_i represents the performance of the sensor. As the value of the factor gets high, the detection probability of event occurrences at the same node gets low. Also, by the definition of $P_j(\overline{\mathscr{X}}) = 1 - \prod_{x \in \overline{\mathscr{X}}}(1 - p(x,j))$, as λ_i increases, the difference of marginal gain that occurs whenever a strategy is added gets small. That is, $P_j(\overline{\mathscr{X}})$ is close to modular function as λ_i increases. By the definition of approximation ratios R, C, and H, the ratios get close to 1 depending on how close the objective function is to a modular function. In these

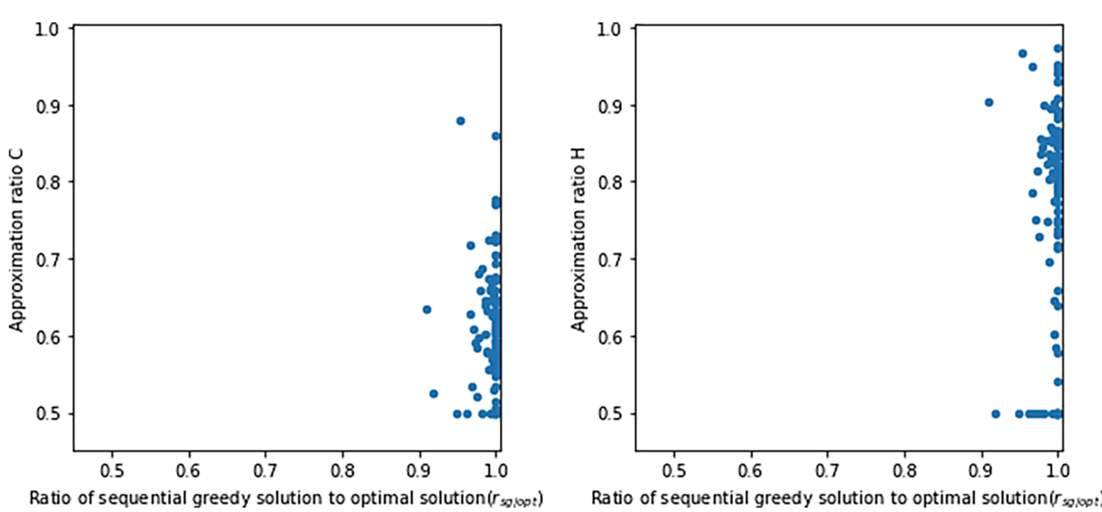

Fig. 3. Comparison between two approximation ratios C and H ($\lambda_i = 0.3$).

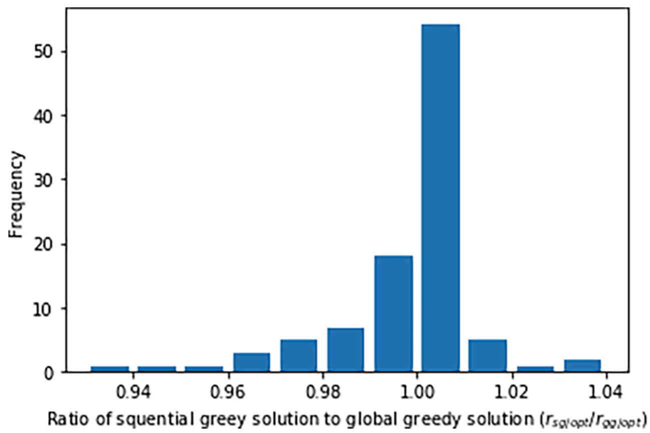

Fig. 4. The ratio of sequential greedy to global greedy ($\lambda_i = 0.3$).

experiments, approximation ratio H was the largest approximation ratio, however, the ranking may change depending on the certain instance situations.

When we set the sensing decay parameter at $\lambda_i = 0.3$, average ratios and computation times for the two algorithms performed well in general. Therefore, We used $\lambda_i = 0.3$ in these experiments. Fig. 3 represents two plots of which each point shows the ratio of the sequential greedy solution to an optimal solution as compared to approximation ratios C (left) and H (right). Compared to approximation ratio C, approximation ratio H was able to give tighter upper bounds of the solutions obtained by the sequential greedy algorithm. In 80 out of 100 instances, approximation ratio H were higher than $1-\frac{1}{e}\approx 0.632$, which was proved by using a randomized continuous greedy algorithm (Vondrák, 2008). Also, we observed that the ratios of the sequential greedy solution to an optimal solution were close to 1 in most cases. This implies that it is important to design improved instance dependent guarantees.

Fig. 4 shows the ratio of the sequential greedy solution to the global greedy solution. Most cases were within the interval [0.97, 1.02]. The performance of the global greedy algorithm was slightly better, but showed almost similar performances. As mentioned before, the computation times of the sequential greedy algorithm were about half of the computation times of the global greedy algorithm.

4.2. Experiments in large data sets

For the large data sets, we set $\lambda_i = 0.3, AL_i \leqslant 6, M$ (10 to 70), and N (20 to 140). Agents and nodes are in $[0,50]^2$. We executed 10 runs for each M and N. Because optimal solutions were intractable to compute within a reasonable time, we used an upper bound instead, which is derived from a relaxed version of the original problem. The way to calculate the upper bound of the problem is as follows: We assume that there is an optimal strategy for each agent without considering other agents. It means that we can calculate $\overline{\overline{\mathscr{X}}}_i = argmax_{k\in\mathscr{X}_i} f(\{k\})$ for each agent. $\sum_{i=1}^{M} f(\overline{\overline{\mathscr{X}}}_i)$ can be an upper bound of the problem. By the definition of $\overline{\overline{\mathscr{X}}}_i$, we know $\sum_{i=1}^{M} f(\overline{\overline{\mathscr{X}}}_i) \geqslant \sum_{i=1}^{M} f(O_i)$. We also know $\sum_{i=1}^{M} f(O_i) \geqslant f(\mathscr{X}^*)$ because function f is submodular. Therefore, we use $\sum_{i=1}^{M} f(\overline{\overline{\mathscr{X}}}_i)$ as an upper bound of the problem. Using the upper bound, the results for the large data sets are summarized in Table 2. $r_{gg/ub}$ and $r_{sg/ub}$ are ratios of the global and sequential greedy algorithm solutions to the upper bound, respectively. The values in Table 2 are average ratios and computation times from the numerical experiments. We excluded the results of approximation ratio R in these experiments because approximation ratio R tended to be lower than approximation ratios C and H. Even though approximation ratio R was lower than approximation ratios C and H in the experiments,

Table 2
Results for the different M, N values.

M	N	$r_{gg/ub}$	$r_{sg/ub}$	C	H	t_{gg} (sec)	t_{sg} (sec)
10	20	0.787	0.792	0.562	0.692	1.655	0.533
20	40	0.773	0.765	0.548	0.757	23.028	3.517
30	60	0.796	0.784	0.531	0.776	109.313	12.623
40	80	0.821	0.808	0.514	0.703	357.825	27.980
50	100	0.850	0.832	0.515	0.717	567.010	30.739
60	120	0.878	0.868	0.511	0.752	1732.029	93.029
70	140	0.907	0.892	0.509	0.761	2267.131	82.271

approximation ratio R might be higher than approximation ratio C or H depending on instances or problems.

The more complex a situation was (as M and N increased), the higher $r_{gg/ub}$ and $r_{sg/ub}$ from the two greedy algorithms in the numerical experiments were obtained. Overall, there was no significant difference in the objective value obtained by the sequential greedy algorithm and the global greedy algorithm under any circumstances. However, when it comes to the computation time, the sequential greedy algorithm was much faster than the global greedy algorithm, as shown in Fig. 5. Fig. 5 gives insights into the difference between the two algorithms in terms of computation times. The important point is that the sequential greedy algorithm can derive the solution within a reasonable time, in complex situations. In particular, the difference in computation time occurs more than 25 times in the case of $M = 70$ and $N = 140$.

We set a specific case in which the number of nodes is fixed to 100 and the number of agents is changed. In reality, the number of nodes is fixed in a specific area, and managers decide the number of agents by considering operating costs, legal issues, and other factors. The results are summarized in Table 3. The values in Table 3 are average ratios and computation times from the numerical experiments. Even when the number of agents is large (i.e., 70 agents), the sequential greedy algorithm obtained solutions within 100 s, and the difference with the upper bound was also within 10%. In particular, the performance difference with the global greedy algorithm was 1.4%, which leads to relatively competitive solutions, when considering the difference between 2,331 s and 99 s. We have confirmed that our approximation ratio H finds a relatively higher ratio than existing approximation ratio C through the numerical experiments. Consequently, we can exploit the largest value of approximation ratios R, C, and H as the bounds of the approximation ratio.

Fig. 6 shows the ratio of the sequential greedy solution to the upper bound ($\sum_{i=1}^{M} f(\overline{\overline{\mathscr{X}}}_i)$), compared with approximation ratio H. In these experiments, when the number of agents was small, it would be better to use approximation ratio H to estimate the upper bound of the problem. On the other hand, as the number of agents increased, it would be better

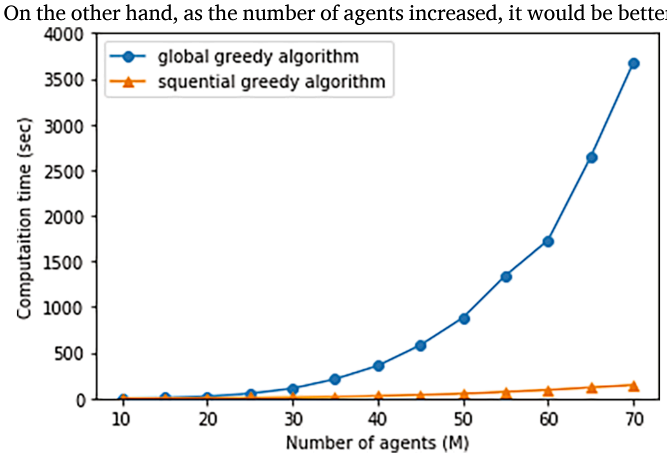

Fig. 5. Computation time of the two greedy algorithms.

Table 3
Results for the different M values with the fixed N value.

M	N	$r_{gg/ub}$	$r_{sg/ub}$	C	H	t_{gg} (sec)	t_{sg} (sec)
10	100	0.600	0.599	0.719	0.930	9.303	2.269
20	100	0.676	0.671	0.593	0.891	62.874	9.077
30	100	0.729	0.726	0.538	0.819	182.240	18.927
40	100	0.798	0.789	0.522	0.762	480.369	34.815
50	100	0.851	0.839	0.515	0.712	818.675	52.828
60	100	0.887	0.877	0.507	0.642	1467.824	73.618
70	100	0.922	0.908	0.504	0.609	2331.036	99.289

to use $\sum_{i=1}^{M} f(\overline{\overline{\mathscr{X}}}_i)$ as an upper bound instead of using approximation ratio H.

The reason for this difference is that, as the number of agents increases, the total number of strategies for the agents increases. According to the definitions of C_1 and C_2, the values of C_1 and C_2 tend to increase as the number of strategies increases. In particular, because the value of C_1 tends to increase more than the value of C_2, the value of ratio H tends to decrease as the number of strategies increases. When it comes to the ratio $r_{sg/ub}$, the ratio increases as the number of agents increases. Given a fixed space, as the number of agents increases, it is more likely to obtain high-quality solutions even through the sequential greedy algorithm. Actually, the average area in which each agent needs to be monitored tends to be smaller as the number of agents increases. This means that the number of strategies that each agent should substantially consider tends to decrease as the number of agents increases. Therefore, it is less likely to obtain solutions somewhat different from the optimal solution as the number of agents increases.

5. Conclusions

In this paper, we presented a mobile multi-agent sensing problem, which is one of the submodular maximization problems under a partition matroid constraint. The sequential and global greedy algorithms were used to obtain high-quality solutions. We introduced new instance dependent guarantees to show improved bounds, depending on instances ($\frac{1}{1+h}, R$). Compared to the ratio of *curvature* ($\frac{1}{1+c_1}$), we presented the novelty and validity of the new approximation ratios of the sequential greedy algorithm. Also, compared to the global greedy algorithm, the sequential greedy algorithm showed competitiveness in terms of performances and computation times. Therefore, the sequential greedy algorithm is expected to be useful when we deal with the mobile multi-agent sensing problem.

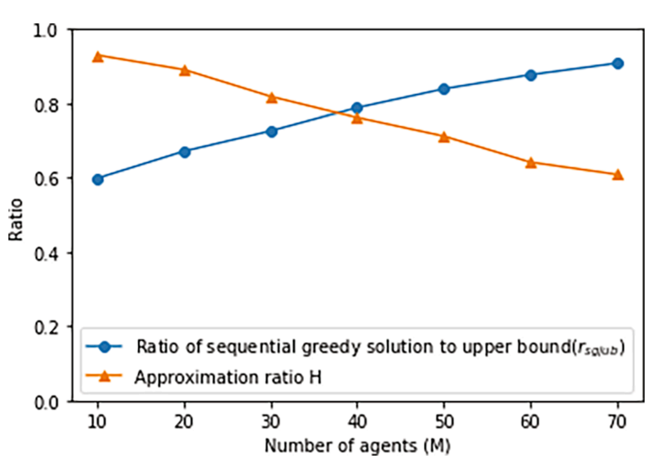

Fig. 6. Approximation ratio H and $r_{sg/ub}$ in the fixed N data sets ($N = 100$).

An important area for future work is to design new algorithms. The algorithms have to prove tighter approximation ratios and take less computation time to obtain high-quality solutions. Furthermore, to obtain optimal (or near-optimal) solutions of the problem efficiently, we need to design exact algorithms such as a cutting-plane algorithm and branch-and-price algorithm. The problems can be extended by considering the uncertainty of the objective function. Applications of stochastic programming to the problem with uncertainty might be future research.

CRediT authorship contribution statement

Jongmin Lee: Conceptualization, Methodology, Investigation, Data curation, Writing - original draft, Writing - review & editing, Visualization. **Gwang Kim:** Conceptualization, Methodology, Software, Data curation, Writing - original draft, Writing - review & editing, Visualization. **Ilkyeong Moon:** Conceptualization, Validation, Writing - review & editing, Supervision.

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