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Robust multiperiod inventory model with a new type of buy one get one promotion: "My Own Refrigerator"☆

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ABSTRACT

Recently, one of the largest retail companies in Korea introduced a mobile application that enables customers to store buy-one-get-one (BOGO) products in their virtual storage for later use. That is, customers who store the extra freebies in their virtual storage can drop by the store to pick them up in the future. Consequently, the application was successful in attracting customers. However, this type of promotion has significant implications for inventory levels. Since customers who buy the product do not need to take both products on the day of purchase, the promotion involves a high degree of uncertainty regarding the revisiting date. To deal with this uncertainty, we propose a robust multiperiod inventory model by addressing the approximation of a multistage stochastic optimization model. Without full information on the distribution, the inventory policy can be derived with support and the first and second moments of uncertainty factors. The presented model is different from previous studies in that the sum of the uncertainty factors in a particular interval is constrained to less than or equal to 1. This part is reformulated as a robust counterpart that retains tractability under modest data sizes. The results of the comparative simulation experiments show that the presented model provides a robust and stable solution against the worst-case scenario. We also obtain managerial insights from the experiments by varying the expiry date according to three types of customers' revisiting tendencies.

1. Introduction

As the globalization of markets accelerates competition among companies, *sales promotion*, which refers to short-term incentives promoting the sale of a product or service, plays a prominent role. Among the various types of sales promotions, the most frequently encountered in daily life is a *price reduction*. For instance, airline companies and hotels reduce the price to promote the sale of remaining seats and rooms, respectively. In the case of a supermarket, the company reduces the prices of perishable foods each day as closing time approaches. A similar tactic over a longer time scale can be observed in the fashion industry, where a company stimulates customer demand through markdowns (clearance sales) at the end of the selling season. Another common promotion is a *buy one get one free* (BOGO) promotion. This promotion looks similar to a price reduction but can be more effective at attracting customers. According to Shampanier et al. [24], customers generally overvalue the benefit of *free* compared to a discounted price. Furthermore, it can undoubtedly reduce stocks further than a price reduction, under the assumption that the same number of customers arrive to purchase.

Under a BOGO promotion, however, customers who want to buy a relatively small quantity of products could be provided with more products than necessary. In the case of customers visiting a convenience store, they might be limited by the weight of the product, the capacity of their refrigerator at home, or the short shelf life of a perishable product. To relieve these limitations while retaining the advantages of the BOGO promotion, *GS Retail*, one of the largest retail companies in Korea, which operates more than 8000 *GS*25 convenience stores, launched the "*My Own Refrigerator*" (MOR) mobile application. Customers who use the MOR application can delay taking the *second product* (*freebie*), put it in their virtual storage, and pick it up another day. This option eliminates the concerns regarding heavy loads, storage capacity, and short product shelf life. As a result, more than ten million users have downloaded the MOR application since GS Retail launched it in March 2011.

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From the standpoint of the retailer, it is possible to amplify customer demand through BOGO promotions with the MOR application. Accordingly, high revenue and customer satisfaction can be expected. However, the retailer still incurs the inventory holding cost even after the products have actually been sold because customers retain the second products in their virtual storage, which corresponds to the actual shelf of the retailer. Thus, these products not only incur holding costs but also occupy capacity. Even if the product remains in the store's inventory, it is a product that has already been sold to the customer and can no longer generate profit. Most of all, there is high uncertainty as to when customers will pick up their second products. Therefore, the retailer must order products taking into account the quantity of the products on the shelf that have already been sold. This suggests that a new method for inventory control is required.

The demand of customers who arrive at the store to buy the promoted products can be estimated based on accumulated historical data over a long period. Concerning the second product, relatively more uncertainty exists as to whether the customers will pick up the product that day or not, or when the customers will revisit the store. Furthermore, if customers who have already purchased products through the MOR application face stockouts when they revisit the store to pick up their second products, brand loyalty could drop sharply. BOGO promotions through MOR can increase customer demand and generate high revenue, but they are also accompanied by a significant potential penalty to brand image. To deal with the uncertainty of revisiting customers, we considered a *robust optimization* approach to the multiperiod inventory model as a countermeasure against the worst-case scenario.

Robust optimization has been actively studied in the context of decision-making under uncertain data. In the robust optimization scheme, input data have been regarded as an uncertain value belonging within a particular range rather than as a nominal value. Robust optimization seeks the optimal solution under the worst-case scenario that guarantees the feasibility of all possible realizations of uncertainty from the input data. For the inventory manager, it is very important to build a flexible and robust model that allows the company to respond to customer demand with a high service level [10]. Soyster [25] first proposed the robust optimization model with a box shape for the uncertainty set. Since then, research has progressed on the structure of the set, providing a less conservative solution while preserving the feasibility guarantee and tractability. Ben-Tal and Nemirovski [5,6] developed the robust optimization model under the uncertainty set in the form of an ellipsoid shape, which provides a less conservative solution. The polyhedral uncertainty set was then developed by Bertsimas and Sim [8]. The box, ellipsoid, and polyhedral shapes have been considered as the standard forms of the uncertainty set in the robust optimization context. Based on the fruitful development of robust optimization theory, various applications have been studied (see [7,14,16]).

Most of the abovementioned studies are based on decision-making in a static situation. A decision made before the realization of all uncertain data is commonly referred to as a *here and now* decision. In contrast, a *wait and see* decision can be applied more naturally in a multistage decision process, based on the partial realization of uncertain data. In the wait and see decision, decision variables are separated by *adjustable variables* and *nonadjustable variables*. Decision variables that are determined after the realization of uncertain data are called adjustable variables. In contrast, decision variables that are determined before the realization of uncertain data are called nonadjustable variables. By partitioning decision variables in this manner, an *adjustable robust optimization* was established [4].

We examined in detail the robust multiperiod inventory problem that is relevant to this study. Bertsimas and Thiele [9] applied a robust optimization to the multiperiod inventory model based on a polyhedral uncertainty set. Although the model was developed based on a here and now decision in the multiperiod setting, the robust counterpart was developed as a tractable linear program. They adopted a budget of uncertainty, proposed by Bertsimas and Sim [8], by forcing the independence of the uncertain data over the period. Ben-Tal et al. [3] adapted the adjustable robust optimization approach to the retailer-supplier flexible commitment contract, which reduces the bullwhip effect by imposing a penalty on a violation of the promised order quantity in advance between the retailer and supplier. By developing the problem as an affinely adjustable robust counterpart with a min-max criterion, they solved the problem against the worst-case scenario efficiently. Subsequently, See and Sim [23] solved the multiperiod inventory problem whose objective function is presented as the expectation under stochastic demand. They considered stochastic demand as a factor-based demand model that is an affine function of the uncertainty factors. In detail, they substituted the objective function as the reasonable upper bound presented by Chen and Sim [13]. By adopting a linear decision rule, the inventory model was developed as a second-order cone problem. Meanwhile, Goh and Sim [15] developed ROME, a software program for solving the robust optimization problems. They also presented three problems: inventory, project crashing, and portfolio selection problems. For the inventory problem, which is the most relevant to this study, they modeled the problem with a constraint that requires satisfaction of the fill rate rather than imposing a penalty cost on stockout inventories. They applied a distributionally robust optimization approach to the fill-rate constraint for all candidates of the distributions. Ang et al. [2] studied a robust storage assignment problem which is operated in the warehouse. They developed stochastic demand as a factor-based demand model and solved the problem with the linear decision rule. Another application in multiperiod inventory control is the empty container repositioning problem. Tsang and Mak [26] and Lee and Moon [19] adapted the linear decision rule to the empty container repositioning problem under uncertain demand. For additional applications of the robust optimization, we refer the reader to the review paper examined by Yanikoglu et al. [29].

As can be seen from the abovementioned studies, the structure of the inventory model depends on the method of modeling the stochastic demand. If the demand of purchasing the BOGO product is restricted to a deterministic value and the demand of revisiting customers who collect the second product is subject to uncertainty, the latter can be developed as an affine function of uncertainty factors. Accordingly, it has the same property in a factor-based demand model. Naturally, we developed a model based on the linear decision rule to deal with the factor-based demand model. In this study, the sum of the uncertainty factors in a particular interval is constrained to less than or equal to 1. This is different from previous studies, which assumed the uncertainty factors as unconstrained random variables for each period. Also, uncertainty factors considered in this study are not zero-mean random variables. To distinguish the characteristics of this study from the previous research, we summarize the relevant literature in Table 1.

From the perspective of the application and modeling, the main contributions of this study are as follows:

- We applied the robust optimization approach to deal with uncertainty in operating the real-world mobile application. Through various experiments, managerial insights were identified that would be helpful to the retailer.
- Previous studies considering the factor-based demand model have mostly assumed zero-mean and unconstrained random variables. In this study, the non zero-mean random variable is considered and the sum of uncertainty factors over the

Table 1
Comparisons of this research and previous relevant studies.

Authors (year)	Uncertain demand	Decision-making policy	Mean and support of uncertainty factor	Constraint of uncertainty factor
Ben-Tal et al. (2004) [4]	Box	LDR[1]	N/A[2]	N/A
Ben-Tal et al. (2005) [3]	Box & Ellipsoid	LDR	N/A	N/A
Bertsimas and Thiele (2006) [9]	Polyhedron	Base stock policy	N/A	N/A
Ben-Tal et al. (2009) [1]	Box	LDR	N/A	N/A
Wei et al. (2011) [28]	Polyhedron	LDR	N/A	N/A
See and Sim (2010) [23]	Factor-based demand model	LDR	Zero mean and bounded support $[-\underline{z}, \bar{z}]$*	Unconstrained
Goh and Sim (2011) [15]	Factor-based demand model	LDR	Non-zero mean and positive bounded support $[0, \bar{z}]$	Unconstrained
Ang et al. (2012) [2]	Factor-based demand model	LDR	Zero mean and bounded support $[-\underline{z}, \bar{z}]$	Unconstrained
Tsang and Mak (2015) [26]	Factor-based demand model	LDR	Zero mean and bounded support $[-\underline{z}, \bar{z}]$	Unconstrained
Lee and Moon (2019) [19]	Factor-based demand model	LDR	Zero mean and bounded support $[-\underline{z}, \bar{z}]$	Unconstrained
This research	Factor-based demand model	LDR	Non-zero mean and positive bounded support $[0, \bar{z}]$	Constrained

LDR[1] and N/A[2] indicate the linear decision rule and not applicable, respectively. *For the bounded support, $\underline{z}$ and $\bar{z}$ are positive.

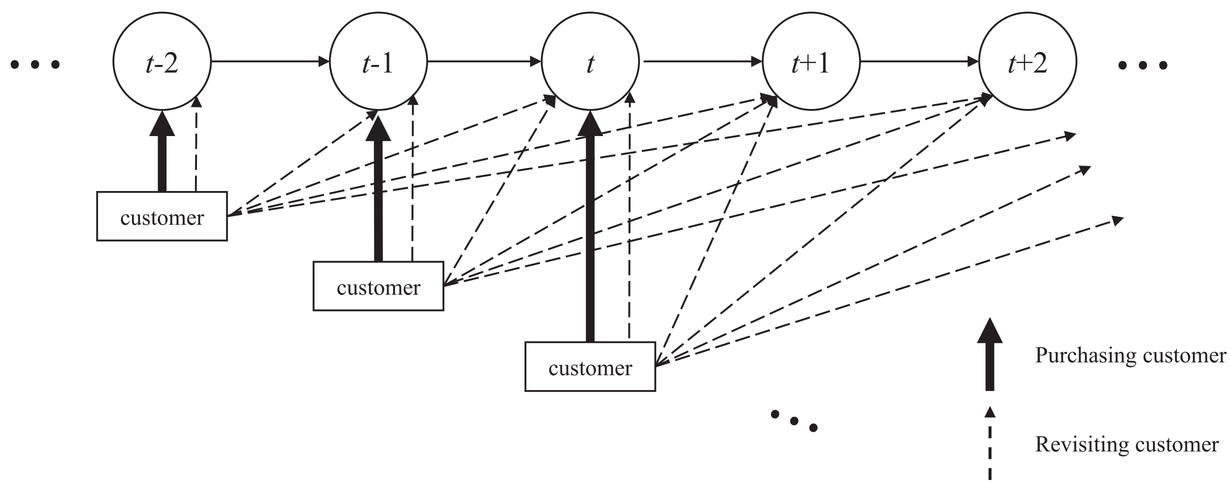

Fig. 1. Two types of demands for each period.

periods is constrained. We developed a robust counterpart that incorporates these features and described the process in detail.

The remainder of the paper is organized as follows: We introduce the problem description of the inventory model for My Own Refrigerator (IMMOR) in Section 2. Section 3 deals with the mathematical formulation of the IMMOR. In Section 4, we present computational experiments and analyses. In Section 5, we summarize the findings of this research.

2. Problem description

We consider a single-item multiperiod inventory model based on the discrete time planning horizon $t \in \{1, \ldots, T\}$. We will use the term *purchasing demand* as the demand of the customer who visits the store to buy the BOGO product through the MOR application. For the demand of the customer who has already made a payment and drops by the store to pick up a second product, we will use the term *revisiting demand*. Each customer can take both products at once or take one and revisit in the future to pick up the second product. These two types of demand for each period are illustrated in Fig. 1. In practice, purchasing demand is more predictable than revisiting demand because historical data for the former have accumulated over a long period. Revisiting demand is less predictable since there is relatively little accumulated data. We consider purchasing demand as a deterministic demand and revisiting demand as a stochastic demand. For this study, we made the following assumptions:

Assumption 1. Customers purchase one package of a promotion product (a set of two products) and take either one or both products at the time of purchase.

Assumption 2. It is unknown when the customers will revisit the store to pick up the second product, but they will revisit before the expiry date from the purchasing date $[t, t+\tau)$.

Assumption 3. Purchasing demand is the deterministic demand, and revisiting demand is the stochastic demand.

Assumption 4. The revisiting rate in the last period $(t = T-1)$ has the value of 1. In other words, customers who buy the BOGO product in the last period take two products because they know that they cannot take the second product in the future.

Assumption 5. The BOGO promotion through the MOR application is valid for a given planning horizon. That is, we assume that it is available from $t = 1$ and ends without salvage value after $t = T$.

The assumptions in this study are made based on the operation of MOR in practice. For more information about the application, we refer readers to the *App Store, Google Play*, or the website of GS25 (http://gs25.gsretail.com). Throughout the paper, we define $\mathfrak{T} \triangleq \{1, \ldots, T\}$ and $\mathfrak{T}^- \triangleq \{1, \ldots, T-1\}$ for brevity in expressing the planning horizon.

2.1. Demand modeling

Let d_t and $\tilde{\xi}_t$ denote the deterministic purchasing demand and stochastic revisiting demand, respectively. Denote by $\tilde{\boldsymbol{\rho}}^t \triangleq (\tilde{\rho}^t_t, \tilde{\rho}^t_{t-1}, \ldots, \tilde{\rho}^t_{t-\tau+1})$ a vector of the *revisiting rate*, where τ is an expiry date from the purchase date. Each revisiting rate means the probability of taking both products at period t, the probability of revisiting to collect the second product at period t from the period $t-1$, …, the probability of revisiting to collect the second product at period t from the period $t-\tau+1$. A set of vectors can be

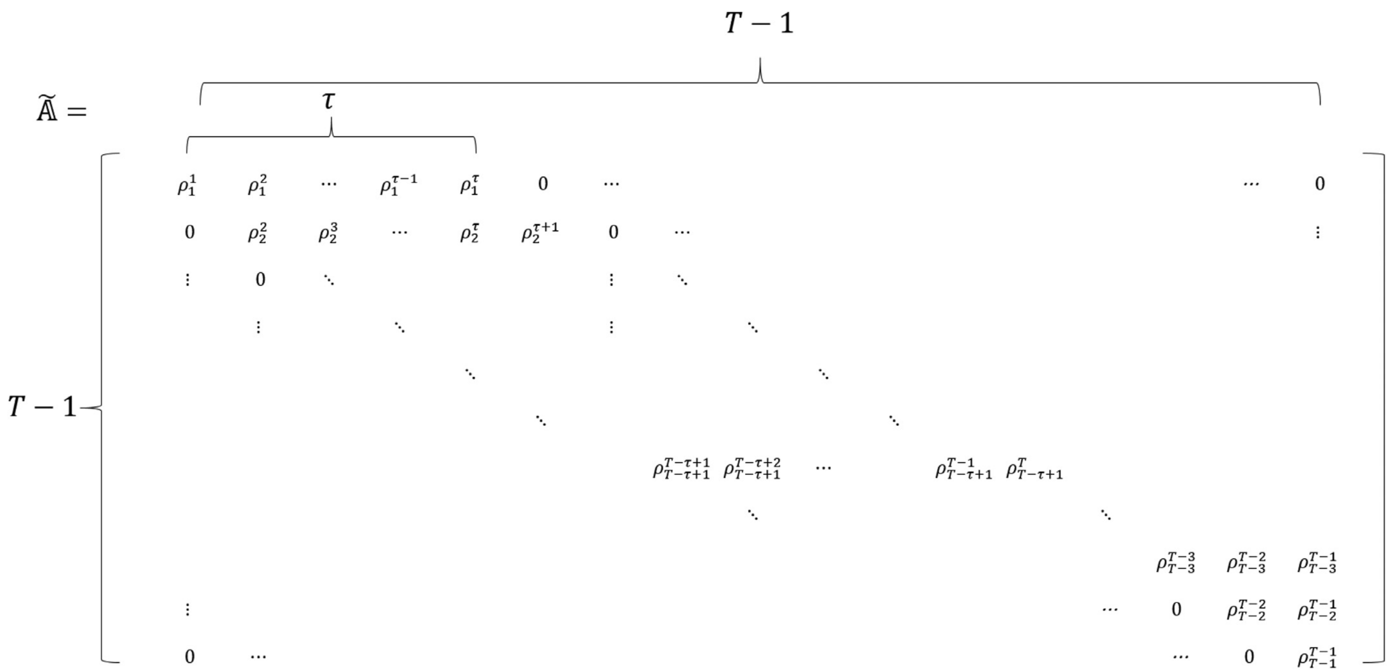

Fig. 2. Matrix of uncertainty factors representing the revisiting rates.

represented by a matrix $\tilde{\mathbb{A}}$, which is illustrated in Fig. 2. We assume that d_t occurs in period $t \in \mathfrak{T}^-$ and it is scattered by a vector of uncertainty factor $\tilde{\boldsymbol{\rho}}^t$. Since each uncertainty factor follows the probability distribution, it has a value between 0 and 1. Also, the sum of the probabilities from t to $t+\tau-1$ is less than or equal to 1. The sum of these probabilities can be 1, but there is no guarantee that all customers will pick up their second products. Therefore, we set the sum to less than or equal to 1. Revisiting demand $\tilde{\xi}_t$ can be modeled as an affine function of uncertainty factors $\tilde{\boldsymbol{\rho}}^t$ as follows:

$$\tilde{\xi}_t(\tilde{\boldsymbol{\rho}}) \triangleq \sum_{i=\max(1,t-\tau+1)}^{t} d_i \tilde{\rho}_i^t \tag{1}$$

$$\text{where} \begin{cases} \displaystyle\sum_{i=t}^{\min(t+\tau-1,T-1)} \tilde{\rho}_t^i \le 1 \\ 0 \le \tilde{\rho}_t^i \le 1 \qquad i \in \{t, \ldots, \min(t+\tau-1, T-1)\} \end{cases} \tag{2}$$

Each revisiting demand $\tilde{\boldsymbol{\rho}}^t$ in period $t \in \mathfrak{T}^-$ is constrained by (2). The parameters related to the demand are summarized as follows:

- d_t Deterministic purchasing demand in period t
- $\mathbf{d_t}$ A vector of purchasing demands from period 1 to $T-1$, that is, $\mathbf{d_t} \triangleq (d_1, d_2, \ldots, d_{T-1})$
- $\tilde{\rho}_i^t$ Revisiting rate from period i to t which is an unknown coefficient
- $\tilde{\boldsymbol{\rho}}^t$ A vector of revisiting rates from period t to $t-\tau+1$
- $\tilde{\xi}_t$ Stochastic demand, which is the aggregated demand of revisiting demands in period t
- $\tilde{\boldsymbol{\xi}}_t$ A vector of stochastic demands from period 1 to $T-1$, that is, $\tilde{\boldsymbol{\xi}}_t \triangleq \mathbf{d_t'}\tilde{\mathbb{A}}$ and $\tilde{\boldsymbol{\xi}}_t = (\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_{T-1})$

2.2. Sequences of the ordering decision

The inventory manager observes the inventory level at the beginning of each period and determines the order quantity to respond to future demand. We assume backlog for understocked inventory. Accordingly, the balance equation (flow conservation) among inventory level, order quantity, and demand is satisfied in each period. Also, we assume that order quantity cannot exceed an upper limit for each period. The main objective is to identify a decision that minimizes the total cost of the planning horizon $t \in \mathfrak{T}$ while satisfying the balance equation and capacity of order quantity. Without loss of generality, we assume the lead time of replenishment as 0. That is, if the product is ordered at the beginning of the period t, it is replenished in the inventory just prior to the beginning of the period $t+1$. In a given planning horizon, the order can be placed until $t = T-1$, and the salvage value of the inventory level is 0 from the period T. The sequence of decision-making in the planning horizon is illustrated in Fig. 3.

3. Mathematical formulation of the IMMOR

We considered two types of decision variables to respond to purchasing demand and revisiting demand. The decision variables x_t and y_t represent the order quantity to satisfy the demand d_t and $\tilde{\xi}_t$, respectively. The inventory manager determines the order quantities x_t and y_t from period $t=1$ to period $t=T-1$. For each order, a unit purchasing cost c_t occurs for x_t and y_t because they are the order quantities for the same item. We assume that backlogging for each inventory level is allowed. Accordingly, the inventory levels for each demand are represented by u_t and v_t, respectively, where $t \in \mathfrak{T}$. If there is overstock (understock) at the end of each period, a unit inventory holding (backlog) cost occurs for each product. It is assumed that the same unit inventory holding cost h_t occurs for the positive values of u_t and v_t. For the negative values of u_t and v_t, different unit backlog costs, b_t and p_t, respectively, are assumed. We consider two types of the unit backlog cost ($b_t \ll p_t$) because the understocking revisiting demand is assumed to affect the brand image, which incurs a significant opportunity cost. Considering the capacity of the order quantity, the sum of x_t and y_t is restricted to an upper limit C_t in each period. Balance equations for purchasing demand and revisiting demand are illustrated in Figs. 4 and 5, respectively.

If the order quantity and inventory level are managed by one type of decision variable, it is difficult to figure out which demand is not satisfied. In addition, a preferential response to revisiting de-

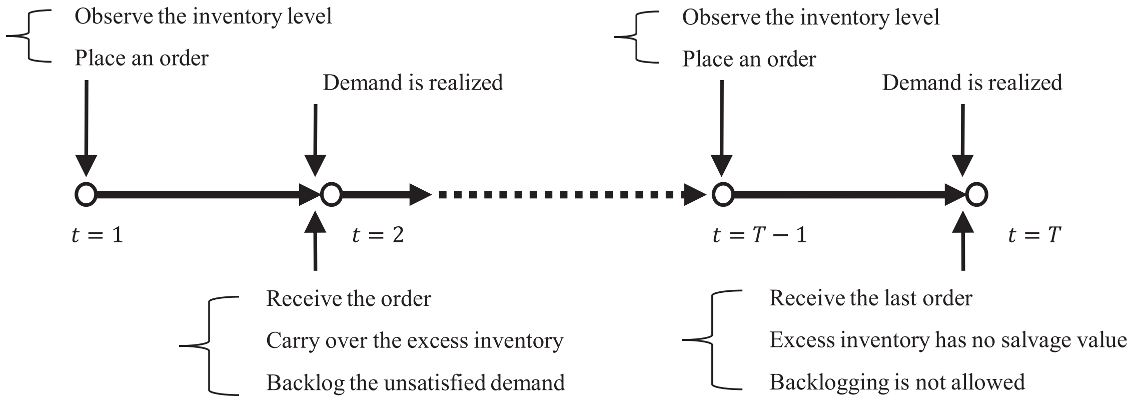

Fig. 3. Sequences of decision making in the planning horizon.

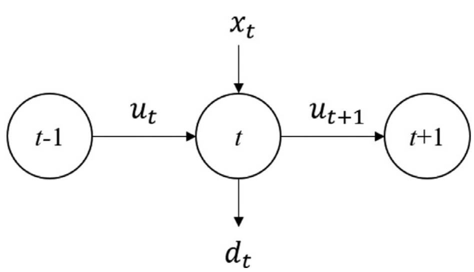

Fig. 4. Balance equation related to purchasing demand.

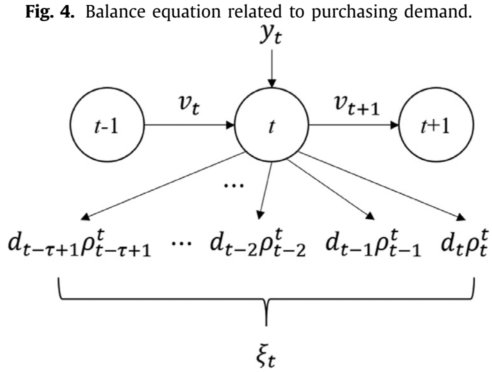

Fig. 5. Balance equation related to revisiting demand.

mand is required. By partitioning the decision variables for order quantity and inventory level to the two types of decision variables (x_t, y_t, and u_t, v_t), and limiting the total order quantity by assigning an enormous backlog cost to the stockout of revisiting demand, the abovementioned issues can be handled. The inventory manager will replace the order by giving priority to revisiting demand. Meanwhile, a fixed cost for replenishment can be considered, and the decision variables can be regarded as integer values. Various types of costs, such as remanufacturing, carbon emissions, defective items, or supplier selection, could be considered to make an inventory model more realistic [11,12,21,22,27]. In this study, however, we formulated a mathematical model as a linear program including purchasing, inventory holding, and backlog costs to retain tractability in the robust optimization approach.

3.1. Mathematical formulation of the IMMOR under deterministic demand

In this section, we present a mathematical formulation based on a linear program under deterministic demand. Before making the order decision for the entire period, the inventory manager regards the uncertainty factor as a deterministic value. Consequently, revisiting demand is also considered as a deterministic value. The mathematical formulation under deterministic demand can be developed as follows:

$$\begin{aligned} \min \quad & \sum_{t\in\mathfrak{T}^-}\Big[c_t(x_t+y_t)+h_t(u_{t+1})^+ + h_t(v_{t+1})^+ \\ & \qquad + b_t(u_{t+1})^- + p_t(v_{t+1})^-\Big] \\ \text{s.t.} \quad & u_{t+1}=u_t+x_t-d_t && t\in\mathfrak{T}^-; \\ & v_{t+1}=v_t+y_t-\xi_t && t\in\mathfrak{T}^-; \\ & x_t+y_t\le C_t && t\in\mathfrak{T}^-; \\ & x_t, y_t\ge 0 && t\in\mathfrak{T}^-; \end{aligned} \tag{3}$$

3.2. Mathematical formulation of the IMMOR under stochastic demand

If revisiting demand is regarded as a random variable as shown in (1), a multistage stochastic optimization model can be considered. In this case, the objective function is expressed as an expectation form $\mathbb{E}(\cdot)$ and all decision variables are affected by uncertainty factors. Accordingly, the multistage stochastic optimization model can be developed as follows:

$$\begin{aligned} \min \quad & \mathbb{E}\Bigg[\sum_{t\in\mathfrak{T}^-}\Big(c_t\big(x_t(\tilde{\boldsymbol{\rho}}^{t-1})+y_t(\tilde{\boldsymbol{\rho}}^{t-1})\big)+h_t\big(u_{t+1}(\tilde{\boldsymbol{\rho}}^{t})\big)^+ \\ & \quad + h_t\big(v_{t+1}(\tilde{\boldsymbol{\rho}}^{t})\big)^+ + b_t\big(u_{t+1}(\tilde{\boldsymbol{\rho}}^{t})\big)^- + p_t\big(v_{t+1}(\tilde{\boldsymbol{\rho}}^{t})\big)^-\Big)\Bigg] \\ \text{s.t.} \quad & v_{t+1}(\tilde{\boldsymbol{\rho}}^{t})=v_t(\tilde{\boldsymbol{\rho}}^{t-1})+y_t(\tilde{\boldsymbol{\rho}}^{t-1})-d_t && t\in\mathfrak{T}^-; \\ & v_{t+1}(\tilde{\boldsymbol{\rho}}^{t})=v_t(\tilde{\boldsymbol{\rho}}^{t-1})+y_t(\tilde{\boldsymbol{\rho}}^{t-1})-\tilde{\xi}_t(\tilde{\boldsymbol{\rho}}^{t}) && t\in\mathfrak{T}^-; \\ & x_t(\tilde{\boldsymbol{\rho}}^{t-1})+y_t(\tilde{\boldsymbol{\rho}}^{t-1})\le C_t && t\in\mathfrak{T}^-; \\ & x_t(\tilde{\boldsymbol{\rho}}^{t-1}), y_t(\tilde{\boldsymbol{\rho}}^{t-1})\ge 0 && t\in\mathfrak{T}^-; \end{aligned} \tag{4}$$

3.3. Robust optimization approach for the IMMOR

In practice, it is difficult to obtain full information on random demand, such as what distribution it follows. Even if the distribution is estimated, evaluating the multistage expectation is intractable. Therefore, instead of directly minimizing the expectation of the objective function, we focused on minimizing the approximated upper bound of the function. By using the linear decision rule, the upper bound of the objective function can be obtained without considering the expected cost function.

The most common form of the factor-based demand model in the robust optimization context is as follows:

$$d_t(\tilde{\mathbf{z}}_{t-1}) \triangleq d_t^0 + \sum_{k=1}^{N} d_t^k \tilde{z}_k \tag{5}$$

where $1 \le N_1 \le N_2 \le \cdots \le N_{T-1} = N$ and the predefined uncertainty factors, $\tilde{\mathbf{z}}_k$, are unfolded until $k = 1, \cdots, N_t$.

Recall that stochastic demand (1) is also an affine function of uncertainty factors $\tilde{\boldsymbol{\rho}}$. It indicates that the demand model (1) can be interpreted as a special case of the factor-based demand model (5). Accordingly, we used stochastic demand (1) as the factor-based demand model.

3.3.1. Linear decision rule

To solve the inventory problem under the factor-based demand model, we adopted the *linear decision rule* (for the sake of brevity, we will hereafter use the abbreviation “LDR”). By restricting decision variables as affinely dependent on the uncertainty factors, the inventory manager can delay the decision by observing the realization of part of the uncertainty factors. Let $x_t{}^{\text{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1})$ and $y_t{}^{\text{LDR}}(\tilde{\boldsymbol{\rho}}^{t-1})$ denote the order decisions based on the LDR as follows :

$$x_t{}^{\text{LDR}}\left(\tilde{\boldsymbol{\rho}}^{t-1}\right) = x_t^0 + \mathbf{x}_t{}'\tilde{\boldsymbol{\rho}}^{t-1}$$
$$y_t{}^{\text{LDR}}\left(\tilde{\boldsymbol{\rho}}^{t-1}\right) = y_t^0 + \mathbf{y}_t{}'\tilde{\boldsymbol{\rho}}^{t-1}$$

Since the decision is based on the realized uncertainty factors, which is referred to as the *non − anticipative* property, we restricted the uncertainty factors that are unavailable in period t. It can be incorporated by summing $x_t^{i,j}\tilde{\rho}_i^j$ and $y_t^{i,j}\tilde{\rho}_i^j$ until $(i, j) \in \{(i, j)|i : i \le j, j : j \le t-1\}$. For brevity in representing the indices, let $\mathcal{N}_j \triangleq \{i \mid i \le j\}$ and $\mathcal{M}_t \triangleq \{j | j \le t-1\}$. Based on the LDR, the order quantity for purchasing demand in each period is expressed as follows:

$$x_t{}^{\text{LDR}}\left(\tilde{\boldsymbol{\rho}}^{t-1}\right) = x_t^0 + \sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j} x_t^{i,j}\tilde{\rho}_i^j \tag{6}$$

The decision based on the LDR corresponding to the order quantity for the revisiting demand can also be expressed as follows:

$$y_t{}^{\text{LDR}}\left(\tilde{\boldsymbol{\rho}}^{t-1}\right) = y_t^0 + \sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j} y_t^{i,j}\tilde{\rho}_i^j \tag{7}$$

That is, the order decision is based on the observed information available at the beginning of the period t.

Remark 1. The inventory levels u_{t+1} and v_{t+1} also take an affine structure with respect to $\tilde{\boldsymbol{\rho}}^t$ as follows:

$$u_{t+1}\left(\tilde{\boldsymbol{\rho}}^t\right) = u_{t+1}^0 + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u_{t+1}^{i,j}\tilde{\rho}_i^j \tag{8}$$

$$v_{t+1}\left(\tilde{\boldsymbol{\rho}}^t\right) = v_{t+1}^0 + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} v_{t+1}^{i,j}\tilde{\rho}_i^j \tag{9}$$

It is easy to show that inventory levels in (8) and (9) also feature the affine function of the uncertainty factors $\tilde{\boldsymbol{\rho}}^t$. This function can be derived with the closed-form expression of the balance equations as follows:

$$u_{t+1}\left(\tilde{\boldsymbol{\rho}}^t\right) = u_1 + \sum_{k=1}^{t} x_k\left(\tilde{\boldsymbol{\rho}}^{k-1}\right) - \sum_{k=1}^{t} d_k$$
$$v_{t+1}\left(\tilde{\boldsymbol{\rho}}^t\right) = v_1 + \sum_{k=1}^{t} x_k\left(\tilde{\boldsymbol{\rho}}^{k-1}\right) - \sum_{k=1}^{t} \tilde{\xi}_k\left(\tilde{\boldsymbol{\rho}}^{k}\right)$$

As a result, the two types of decision variables related to the inventory level also take the non-anticipative property.

3.3.2. Upper bound of the expected positive parts

We assumed that the inventory manager decides on the order quantity based on stochastic demand in the absence of full information. Therefore, minimizing the reasonable upper bound was focused rather than directly minimizing the expectation value of the objective function. As shown in the stochastic optimization model (4), the objective function includes the purchasing, inventory holding, and backlog costs over the entire planning horizon. For the purchasing cost, the expected value can be obtained as follows:

$$\begin{aligned}
&\mathbb{E}\left(c_t\left(x_t\left(\tilde{\boldsymbol{\rho}}^{t-1}\right) + y_t\left(\tilde{\boldsymbol{\rho}}^{t-1}\right)\right)\right)\\
&= \mathbb{E}\left(c_t\left(x_t^0 + \sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j} x_t^{i,j}\tilde{\rho}_i^j + y_t^0 + \sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j} y_t^{i,j}\tilde{\rho}_i^j\right)\right)\\
&= c_t(x_t^0 + y_t^0) + c_t\mathbb{E}\left(\sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j}\left(x_t^{i,j} + y_t^{i,j}\right)\tilde{\rho}_i^j\right)\\
&= c_t(x_t^0 + y_t^0) + c_t\sum_{j\in\mathcal{M}_t}\sum_{i\in\mathcal{N}_j}\left(x_t^{i,j} + y_t^{i,j}\right)\mu_i^j\\
&\quad \text{where } \mu_i^j = \mathbb{E}[\tilde{\rho}_i^j]
\end{aligned}$$

In the case of the inventory holding cost, we approximated the upper bound of the expectation of the positive parts by adapting the work of Chen and Sim [13], who derived the upper bound based on the following theorem:

Theorem 1. *(Chen and Sim [13]) If uncertainty factors are zero-mean random variables with the positive definite covariance matrix under the support set* $\mathbf{W}$ *which is second-order conic representable, the upper bound of* $\mathbb{E}((y_0 + \mathbf{y}'\tilde{\mathbf{z}})^+)$*, which is represented by* $\pi(y_0, \mathbf{y})$*, can be obtained through the optimization problem as follows:*

$$\begin{aligned}
\pi(y_0, \mathbf{y}) = \min\quad & r_1 + r_2 + r_3 + r_4 + r_5\\
\text{s.t.}\quad & y_{10} + \max_{\tilde{\mathbf{z}}\in\mathbf{W}} \tilde{\mathbf{z}}'\mathbf{y_1} \le r_1\\
& r_1 \ge 0\\
& \max_{\tilde{\mathbf{z}}\in\mathbf{W}} \tilde{\mathbf{z}}'(-\mathbf{y_2}) \le r_2\\
& y_{20} \le r_2\\
& \frac{1}{2}y_{30} + \frac{1}{2}\left\| y_{30}, \sum^{1/2}\mathbf{y_3}\right\|_2 \le r_3\\
& \inf_{\mu>0} \frac{\mu}{e} exp\left(\frac{y_{40}}{\mu} + \frac{\|\mathbf{u}\|_2^2}{2\mu^2}\right) \le r_4\\
& u_j \ge p_j y_{4j} && j \in \left\{j : p_j < \infty\right\}\\
& y_{4j} \le 0 && j \in \left\{j : p_j = \infty\right\}\\
& u_j \ge -q_j y_{4j} && j \in \left\{j : q_j < \infty\right\}\\
& y_{50} + \inf_{\mu>0} \frac{\mu}{e} exp\left(-\frac{y_{40}}{\mu} + \frac{\|\mathbf{v}\|_2^2}{2\mu^2}\right) \le r_5\\
& v_j \ge q_j y_{5j} && j \in \left\{j : q_j < \infty\right\}\\
& y_{5j} \le 0 && j \in \left\{j : q_j = \infty\right\}\\
& v_j \ge -p_j y_{5j} && j \in \left\{j : p_j < \infty\right\}\\
& y_{10} + y_{20} + y_{30} + y_{40} + y_{50} = y_0\\
& \mathbf{y_1} + \mathbf{y_2} + \mathbf{y_3} + \mathbf{y_4} + \mathbf{y_5} = \mathbf{y}\\
& r_i, y_{i0} \in \mathcal{R}, \quad \mathbf{y}_i \in \mathcal{R}^N, \quad i = 1, \dots, 5\\
& \mathbf{u}, \mathbf{v} \in \mathcal{R}^N
\end{aligned} \tag{10}$$

The most distinctive difference between the model in this research and that of Chen and Sim [13] is the structure of the uncertainty set. In their work, each predefined uncertainty factor $\tilde{\mathbf{z}}$ belongs to $\mathbf{W}$ which can be correlated but unconstrained over the period. In this study, the sum of the uncertainty factors in a particular interval is constrained. Also, uncertainty factors are not zero-mean random variables. The optimization problem (10) was derived based on zero-mean random variables and support set $\mathbf{W}$. However, we derived the upper bound of the expected positive parts based on non zero-mean random variables with support set $\boldsymbol{\Xi}$.

Remark 2. The reasonable upper bound can be obtained without considering the information of the directional deviations which are related r_4 and r_5 ([23]). That is, the upper bound can be achieved even if p_j and q_j are set to ∞.

In this study, we derived the three upper bounds of the expected positive parts related to excess inventories as follows:

$$\mathbb{E}\left((u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\right) \le \left(u^0_{t+1} + \max_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}}\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^+ = \pi^1\left(u^0_{t+1}, \boldsymbol{u}_{t+1}\right)$$

The second upper bound can be derived by using the equality $a^+ = a + (-a)^+$. Recall that the supports of the uncertainty factors are defined in [0, 1]. Accordingly, the value of the expectation is not canceled by zero mean as shown in [13,23].

$$\begin{aligned}
\mathbb{E}\left(\left(u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^+\right) &= \mathbb{E}\left(u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right) \\
+\mathbb{E}\left(\left(-u^0_{2,t+1} - \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^+\right) &= u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} \\
+\mathbb{E}\left(\left(-u^0_{t+1} - \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^+\right) &\le u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} \\
+\mathbb{E}\left(\left(-u^0_{t+1} + \max_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}}\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^+\right) & \\
= u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} & \\
+\left(-u^0_{t+1} + \max_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}}\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^+ &= \pi^2\left(u^0_{t+1}, \boldsymbol{u}_{t+1}\right)
\end{aligned}$$

The third upper bound can be derived by using the equality $a^+ = (a + |a|)/2$. As with the second upper bound, the value of the expectation is not canceled by zero mean, as shown in the following:

$$\begin{aligned}
\mathbb{E}\left((u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i)^+\right) &= \frac{1}{2}\mathbb{E}\left(u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right) \\
&+\frac{1}{2}\mathbb{E}\left|u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right| \\
&\le \frac{1}{2}u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \frac{1}{2}\sqrt{\mathbb{E}\left[\left(u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^2\right]} \\
&= \frac{1}{2}u^0_{t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} \\
&+\frac{1}{2}\sqrt{(u^0_{t+1})^2 + 2u^0_{t+1}\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \mathbb{E}\left(\left(\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^2\right)} \\
&= \pi^3(u^0_{t+1}, \boldsymbol{u}_{t+1})
\end{aligned}$$

where $\mathbb{E}\left(\left(\sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} u^{i,j}_{t+1}\tilde{\rho}^j_i\right)^2\right)$

$$= \sum_{j,k\in\mathcal{M}_{t+1}}\sum_{i,l\in\mathcal{N}_j}\left((u^{i,j}_{t+1})^2\left(\sigma_{\tilde{\rho}^j_i\tilde{\rho}^j_i} + (\mu^j_i)^2\right) + 2u^{i,j}_{t+1}u^{l,k}_{t+1}\left(\sigma_{\tilde{\rho}^j_i\tilde{\rho}^k_l} + \mu^j_i\mu^k_l\right) + (u^{l,k}_{t+1})^2\left(\sigma_{\tilde{\rho}^k_l\tilde{\rho}^k_l} + (\mu^k_l)^2\right)\right)$$

and σ indicates the covariance of the uncertainty factors.

By minimizing the three bounds, $\pi^1(u^0_{t+1}, \boldsymbol{u}_{t+1})$, $\pi^2(u^0_{t+1}, \boldsymbol{u}_{t+1})$, and $\pi^3(u^0_{t+1}, \boldsymbol{u}_{t+1})$, in the following optimization problem (11), the tightest upper bound $\pi(u^0_{t+1}, \boldsymbol{u}_{t+1})$ can be obtained.

$$\begin{aligned}
\pi\left(u^0_{t+1}, \boldsymbol{u}_{t+1}\right) \triangleq \min &\sum_{i=1}^{3}\pi^i\left(u^0_{i,t+1}, \boldsymbol{u}_{i,t+1}\right) \\
\text{s.t.} \quad &\sum_{i=1}^{3} u^0_{i,t+1} = u^0_{t+1} \\
&\sum_{i=1}^{3} \boldsymbol{u}_{i,t+1} = \boldsymbol{u}_{t+1}
\end{aligned} \tag{11}$$

To retain tractability in solving the optimization problem (11), Assumption A is required. Otherwise, both the problem (11) and the robust optimization model become intractable.

Assumption A. Uncertainty factors $\tilde{\boldsymbol{\rho}}$ representing the revisiting rates are the random variables distributed in the particular intervals as presented in (2). Although the distribution is not known, each uncertainty factor $\tilde{\boldsymbol{\rho}}$ lies in a support set $\boldsymbol{\Xi}$, which is a polyhedron, as shown in (2).

By adopting the work of Chen and Sim [13], the optimization problem (11) for every tth period ($t \in \mathfrak{T}^-$) can be expressed as the epigraph form as follows:

$$\begin{aligned}
\pi\left(u^0_{t+1}, \boldsymbol{u}_{t+1}\right) = \min \quad & r_{1,t+1} + r_{2,t+1} + r_{3,t+1} \\
\text{s.t.} \quad & u^0_{1,t+1} + \max_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}} \tilde{\boldsymbol{\rho}}'\boldsymbol{u}_{1,t+1} \le r_{1,t+1} \\
& r_{1,t+1} \ge 0 \\
& \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} + \max_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}} \tilde{\boldsymbol{\rho}}'(-\boldsymbol{u}_{2,t+1}) \le r_{2,t+1} \\
& u^0_{2,t+1} + \sum_{j\in\mathcal{M}_{t+1}}\sum_{i\in\mathcal{N}_j} \mu^j_i u^{i,j}_{t+1} \le r_{2,t+1} \\
& \frac{1}{2}u^0_{3,t+1} + \frac{1}{2}\left\| u^0_{3,t+1}, \sum^{1/2}\boldsymbol{u}_{3,t+1}\right\|_2 \le r_{3,t+1} \\
& u^0_{1,t+1} + u^0_{2,t+1} + u^0_{3,t+1} = u^0_{t+1} \\
& \boldsymbol{u}_{1,t+1} + \boldsymbol{u}_{2,t+1} + \boldsymbol{u}_{3,t+1} = \boldsymbol{u}_{t+1} \\
& r_{i,t+1}, u^0_{i,t+1} \in \mathcal{R}, \boldsymbol{u}_{i,t+1} \in \mathcal{R}^{T\times T} \\
& i = 1, 2, \text{ and } 3
\end{aligned} \tag{12}$$

According to See and Sim [23], $\pi(\cdot,\cdot)$ in the optimization problem (10) is not exactly second-order cone representable because of the infimum term ($\inf_{\mu>0}\frac{\mu}{e}exp(\cdot)$). However, the infimum term becomes redundant in this model because we assume p_j and q_j as ∞. If the constraints associated with r_1 and r_2, which still contain the uncertainty factors, are well defined as a robust counterpart, the remaining terms associated with the upper bound are all second-order cones. By replacing the max(·) term with the dual linear program, we can derive the robust counterpart. Consider $\max_{\tilde{\boldsymbol{\rho}}\in\boldsymbol{\Xi}}\tilde{\boldsymbol{\rho}}'\boldsymbol{u}_{1,t+1}$ in the first constraint. As shown in (2), uncertainty factors $\tilde{\boldsymbol{\rho}}^t$ in this model feature the polyhedron structure.

For every period, $t \in \mathfrak{T}^-$, we have the following inner optimization problem:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} u_{t+1}^{i,j} \tilde{\rho}_i^j \\ \text{s.t.} \quad & \sum_{j=i}^{\min(i+\tau-1,t)} \tilde{\rho}_i^j \le 1 \qquad && i \in \{1, \cdots, t\}; \\ & 0 \le \tilde{\rho}_i^j \le 1 && i \in \mathcal{N}_j, j \in \mathcal{M}_{t+1}; \\ & \tilde{\rho}_i^j = 0 && i \in \{i | i + \tau \le j\}, j \in \{\tau+1, \cdots, t\}; \end{aligned} \tag{13}$$

By strong duality, each inner optimization problem for every period $(t \in \mathfrak{T}^-)$ can be reformulated as a dual linear program as follows:

$$\begin{aligned} \min \quad & \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} \alpha_{t+1}^{i,j} + \sum_{i=1}^{t} \beta_{t+1}^i \\ \text{s.t.} \quad & \alpha_{t+1}^{i,j} + \beta_{t+1}^i \ge u_{t+1}^{i,j} && i \in \mathcal{N}_j, j \in \mathcal{M}_{t+1}; \\ & \alpha_{t+1}^{i,j} \ge 0 && i \in \mathcal{N}_j, j \in \mathcal{M}_{t+1}; \\ & \beta_{t+1}^i \ge 0 && i \in \{i | i \le t\}; \\ & \alpha_{t+1}^{i,j} = 0 && i \in \{i | i + \tau \le j\}, j \in \{\tau+1, \ldots, t\}; \end{aligned} \tag{14}$$

where $\alpha_t^{i,j}$ and β_t^i are the dual variables of each constraint in (13), respectively.

By replacing the max(·) term with the dual linear program presented in (14), the robust counterpart can be achieved. With the same manner, max(·) term in the constraint related to r_2 is also reformulated to the robust counterpart. By solving the robust counterpart of the optimization problem (12), the tighter upper bound can be achieved than that of each bound.

$$\mathbb{E}\left(\left(u_{t+1}^0 + \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} u_{t+1}^{i,j} \tilde{\rho}_i^j\right)^+\right) \le \pi\left(u_{t+1}^0, \boldsymbol{u}_{t+1}\right) \le \min_{i=1,2,3} \pi^i\left(u_{t+1}^0, \boldsymbol{u}_{t+1}\right)$$

The upper bound of the expected costs related to backlogged inventories can be derived similarly as follows:

$$\mathbb{E}\left(\left(u_{t+1}^0 + \sum_{j \in \mathcal{M}_{t+1}} \sum_{i \in \mathcal{N}_j} u_{t+1}^{i,j} \tilde{\rho}_i^j\right)^-\right) \le \pi\left(-u_{t+1}^0, -\boldsymbol{u}_{t+1}\right) \le \min_{i=1,2,3} \pi^i\left(-u_{t+1}^0, -\boldsymbol{u}_{t+1}\right)$$

3.3.3. Robust counterpart of the IMMOR

Based on the LDR, the robust counterpart of the IMMOR (RIMMOR) can be formulated as follows:

$$\begin{aligned} \min \quad & \sum_{t \in \mathfrak{T}^-} \Bigg[c_t\left(x_t^0 + y_t^0\right) + c_t\left(\sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} \left(x_t^{i,j} + y_t^{i,j}\right)\mu_i^j\right) \\ & + h_t \pi\left(u_{t+1}^0, \boldsymbol{u}_{t+1}^{i,j}\right) + h_t \pi\left(v_{t+1}^0, \boldsymbol{v}_{t+1}^{i,j}\right) \\ & + b_t \pi\left(-u_{t+1}^0, -\boldsymbol{u}_{t+1}^{i,j}\right) + p_t \pi\left(-v_{t+1}^0, -\boldsymbol{v}_{t+1}^{i,j}\right) \Bigg] \\ \text{s.t.} \quad & u_{t+1}^0 = u_t^0 + x_t^0 - d_t && t \in \mathfrak{T}^-; \\ & u_{t+1}^{i,j} = u_t^{i,j} + x_t^{i,j} && t \in \mathfrak{T}^-, i \in \mathcal{N}_j, j \in \mathcal{M}_t; \\ & v_{t+1}^0 = v_t^0 + y_t^0 && t \in \mathfrak{T}^-; \\ & u_{t+1}^{i,j} = \begin{cases} -d_i, & t \in \mathfrak{T}^-, j = t, \ j - \tau + 1 \le i \le j; \\ v_t^{i,j} + y_t^{i,j} & t \in \mathfrak{T}^-, i \in \mathcal{N}_j, j \in \mathcal{M}_t; \end{cases} \\ & x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} + y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \le C_t && t \in \mathfrak{T}^-, \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}; \\ & x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} \ge 0 && t \in \mathfrak{T}^-, \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}; \\ & y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \ge 0 && t \in \mathfrak{T}^-, \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}; \end{aligned} \tag{15}$$

Remark 3. RIMMOR does not need non-anticipative constraints such as, $u_{t+1}^{i,j} = 0$, $v_{t+1}^{i,j} = 0$ $(t \in \mathfrak{T}^-,\ i \in \mathcal{N}_j,\ j \in \mathcal{M}_{t+1})$ or $x_t^{i,j} = 0$, $y_t^{i,j} = 0$ $\left(t \in \mathfrak{T}^-, i \in \mathcal{N}_j, j \in \mathcal{M}_t\right)$. Eqs. (6)–(9) already incorporate the non-anticipative property by summing up the decision variables until the available uncertainty factors in each period t.

For the constraint representing the capacity of the order quantity, the uncertainty factors also remain. Therefore, we reformulated the constraint in each period $(t \in \mathfrak{T}^-)$ as the robust counterpart in the same manner from (13) to (14):

$$x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} + y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \le C_t, \qquad \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}$$

$$\iff x_t^0 + y_t^0 + \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}}\left(\sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} \left(x_t^{i,j} + y_t^{i,j}\right)\tilde{\rho}_i^j\right) \le C_t$$

$$\iff \begin{cases} x_t^0 + y_t^0 + \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} \theta_t^{i,j} + \sum_{i=1}^{t-1} \delta_t^i \le C_t & \\ \theta_t^{i,j} + \delta_t^i \ge x_t^{i,j} + y_t^{i,j} & i \in \mathcal{N}_j, j \in \mathcal{M}_t; \\ \theta_t^{i,j} \ge 0 & i \in \mathcal{N}_j, j \in \mathcal{M}_t; \\ \delta_t^i \ge 0 & i \in \{i | i \le t-1\}; \\ \theta_t^{i,j} = 0 & i \in \{i | i + \tau \le j\}, j \in \{\tau+1, \ldots, t-1\}; \end{cases}$$

where $\theta_t^{i,j}$ and δ_t^i are the dual variables.

In the cases of constraints related to non-negative conditions of decision variables, the constraints of all periods $(t \in \mathfrak{T}^-)$ can also be reformulated by defining the inner optimization problems as follows:

$$\begin{cases} x_t^0 + \mathbf{x_t}'\tilde{\boldsymbol{\rho}} \ge 0, & \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi} \\ y_t^0 + \mathbf{y_t}'\tilde{\boldsymbol{\rho}} \ge 0, & \tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi} \end{cases}$$

$$\iff \begin{cases} x_t^0 - \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} x_t^{i,j} \tilde{\rho}_i^j \ge 0 \\ y_t^0 - \max_{\tilde{\boldsymbol{\rho}} \in \boldsymbol{\Xi}} \sum_{j \in \mathcal{M}_t} \sum_{i \in \mathcal{N}_j} y_t^{i,j} \tilde{\rho}_i^j \ge 0 \end{cases}$$

By developing the dual linear program and substituting it for the inner optimization problem, the robust counterpart can be derived. We omit the expression of the robust counterpart which has the same process from (13) to (14). As a result, the deterministic approximated second-order cone program is derived from the multi-stage stochastic optimization model. We provide a small-size numerical example in Appendix A to make it easier for readers to understand.

3.3.4. Relation to the restricted linear decision rule

Recall that Fig. 2 represents the coefficient matrix of the revisiting rate. When the duration of the entire period T is relatively larger than the expiry date τ, most of the coefficients are zero. Accordingly, the inventory balance equation associated with revisiting rates whose values are zero does not have an effect. In other words, the parts where the values of $\tilde{\boldsymbol{\rho}}^t$ are zero do not directly affect the inventory level, leaving only the balance equation between the relevant decision variables. By forcing the decision variables of these parts to zero, the solution space could be reduced, which helps the commercial optimization solver to find a solution efficiently. In this manner, Ang et al. [2] proposed a *restricted linear decision rule* (RLDR).

Proposition 1. *In this model, the objective value obtained by the RLDR provides the same objective value as the LDR, which was known to provide an inferior solution from the robust counterpart model in [2].*

We will support Proposition 1 through an example. Consider the problem with planning horizon $t \in \{1, \ldots, 8\}$ and an expiry date τ as 3. Consequently, some revisiting rates, such as $(\tilde{\rho}_1^4, \tilde{\rho}_1^5, \tilde{\rho}_2^5, \cdots)$, become zero. For simplicity, consider only the balance equation for $\tilde{\rho}_1^4$ at $t = 6$. According to the LDR, the robust counterpart includes the equation $v_6^{1,4} = v_5^{1,4} + y_5^{1,4}$ in the balance equation $v_6^0 + \sum_{j=1}^5 \sum_{i:i \le j} v_6^{i,j} \tilde{\rho}_i^j = v_5^0 + \sum_{j=1}^4 \sum_{i:i \le j} v_5^{i,j} \tilde{\rho}_i^j +$

Table 2
Results of Experiment 1.

	Data (total planning horizon_expiry date) when order capacity is 350							
	20_5	20_10	25_5	25_10	25_15	30_10	35_5	35_8
Objective value	34119.7	39939.3	31678.9	39706.8	45298.7	47253.3	54248.6	62904.0
Computation time (s)	22.9	59.6	71.4	277.5	499.7	866.5	592.4	1434.5

$y_5^0 + \sum_{j=1}^4 \sum_{i:i\leq j} y_5^{i,j}\tilde{\rho}_i^j - d_3\tilde{\rho}_3^5 - d_4\tilde{\rho}_4^5 - d_5\tilde{\rho}_5^5$. As presented in the balance equation, $v_6^{1,4} = v_5^{1,4} + y_5^{1,4}$ does not affect the inventory level. Restricting the relevant decision variables $y_5^{1,4}$ to zero can allow the problem to be solved efficiently while retaining the objective value.

4. Computational experiments

In this section, we describe the results of three types of computational experiments. The experiments were conducted to answer the following research questions:

(i) Does RIMMOR, which constrains the sum of the uncertainty factors over the period to less than or equal to 1, retain tractability until a modest data size, as the model of See and Sim [23] does?
(ii) How much robustness does RIMMOR guarantee when random demand is realized compared to the deterministic model which estimates the uncertainty factors?
(iii) Depending on the propensity of the customer, what tendency does the inventory policy of RIMMOR show?
(iv) What tendency does total cost show when the expiry date varies?

Research questions (i), (ii), and (iii) and (iv), are answered by Experiments (1)–(3), respectively. Results of Experiments (1)–(3) and analyses are described in Section 4.1 – 4.3. All computational experiments were conducted by FICO XPRESS-IVE version 7.2 with an Intel Core^{TM} i5-7400 CPU @ 3.0 GHz.

RIMMOR needs the mean and covariance of the uncertainty factors. Most of the previous studies that considered a factor-based demand model assumed the uncertainty factors as zero-mean random variables and unconstrained. Accordingly, the mean and covariance could be easily derived. In the RIMMOR, however, each uncertainty factor has support between 0 and 1, and the sum of uncertainty factors within a certain interval is less than or equal to 1. This makes deriving an accurate mean and covariance difficult. Therefore, we estimated the mean and covariance through data sampling with 10,000 iterations. Pseudocode for the generation of the uncertainty factors is described in Appendix B. We assume that all customers revisit the store because we want to observe protection against the worst case. Thus, we made the sum to be 1 by forcing the last iteration of Algorithm 1.

4.1. Experiment 1: tractability of the RIMMOR

We conducted Experiment 1 to investigate the tractability of the RIMMOR. Experiment 1 was conducted by varying the planning horizon and expiry date. The sample mean and covariance were estimated from data generated through Algorithm 1. The results of Experiment 1 are presented in Table 2. As can be seen from Table 2, when the planning horizon increased, the computation time also increased. Furthermore, as the expiry date increased, the computation time increased. Nevertheless, the RIMMOR was tractable until a modest data size. In practice, a retailer who runs the MOR application in a convenience store has one order cycle per day. With the RIMMOR, the retailer can establish a one-month plan for the BOGO promotion.

4.2. Experiment 2: robustness of the RIMMOR

The solution obtained through the RIMMOR is a decision rule for an order quantity. Thus, solving the optimization problem does not provide the order quantity for each period but establishes a policy. To figure out how the decision rule guarantees the protection of the realized uncertain data, we conducted comparative experiments. For the comparison group, *simulation*, we assume that the inventory manager regards the uncertainty factors as deterministic values by estimating the mean based on the data from Algorithm 1. Accordingly, uncertain demands were set as deterministic values for the entire period. The order quantities were obtained by solving the deterministic model (2). In this manner, simulation results and policies from the LDR were compared through 10,000 iterations of the experiment. A summary of the results is illustrated in Fig. 6.

As we can see from Fig. 6, the robustness of RIMMOR was guaranteed compared to the simulation experiments. Although the objective values of the LDR were worse than the best case of the deterministic model, the RIMMOR showed overwhelmingly better results for the worst case. The most important thing to recognize is that the RIMMOR provided stable solutions in terms of the fluctuation. Even though uncertainty factors can be realized with any value, the difference between the minimum and maximum objective values was not significant.

4.3. Experiment 3: effect of duration of the expiry date under the different customers' revisiting propensities

We conducted Experiment 3 to explore the effects of the duration of the expiry date and customers' revisiting propensities on the total cost. One of the research questions was how the objective value changes by varying the expiry date. We could make two conflicting inferences at the same time. First, we expected that the total expectation cost would be lowered by the smoothing effect when the expiry date becomes longer. Second, we thought that a larger order quantity should be replenished to cope with the worst-case scenario, which would incur a higher cost. We also thought that customers' revisiting tendencies might affect the total cost. Therefore, Experiment 3 was conducted by varying the expiry date τ according to three types of customers: (i) a *general customer* (GC) who was already considered in the previous subsection; (ii) an *impetuous customer* (IC), who has a high revisiting rate near the purchasing date; and (iii) a *procrastinating customer* (PC), who has a high revisiting rate when the expiry date approaches. For IC, we assumed that the two products are most likely to be taken on the purchasing date and the revisiting rate decreases as the expiry date approaches. In the case of PC, it is assumed that the revisiting rate increases the further the expiry date from the purchasing date. Data generations for IC and PC are described in Algorithms 2 and 3 in Appendix C. Using the generated data, we conducted Experiment 3 to explore how the objective value varies according to the duration of the expiry date and customer type. To demonstrate the validity of the objective value, *expected value given perfect information* (EV|PI) was introduced to substitute the multistage stochastic optimization model (4). Since the model (4) is difficult to solve directly, we solved each of the 10,000

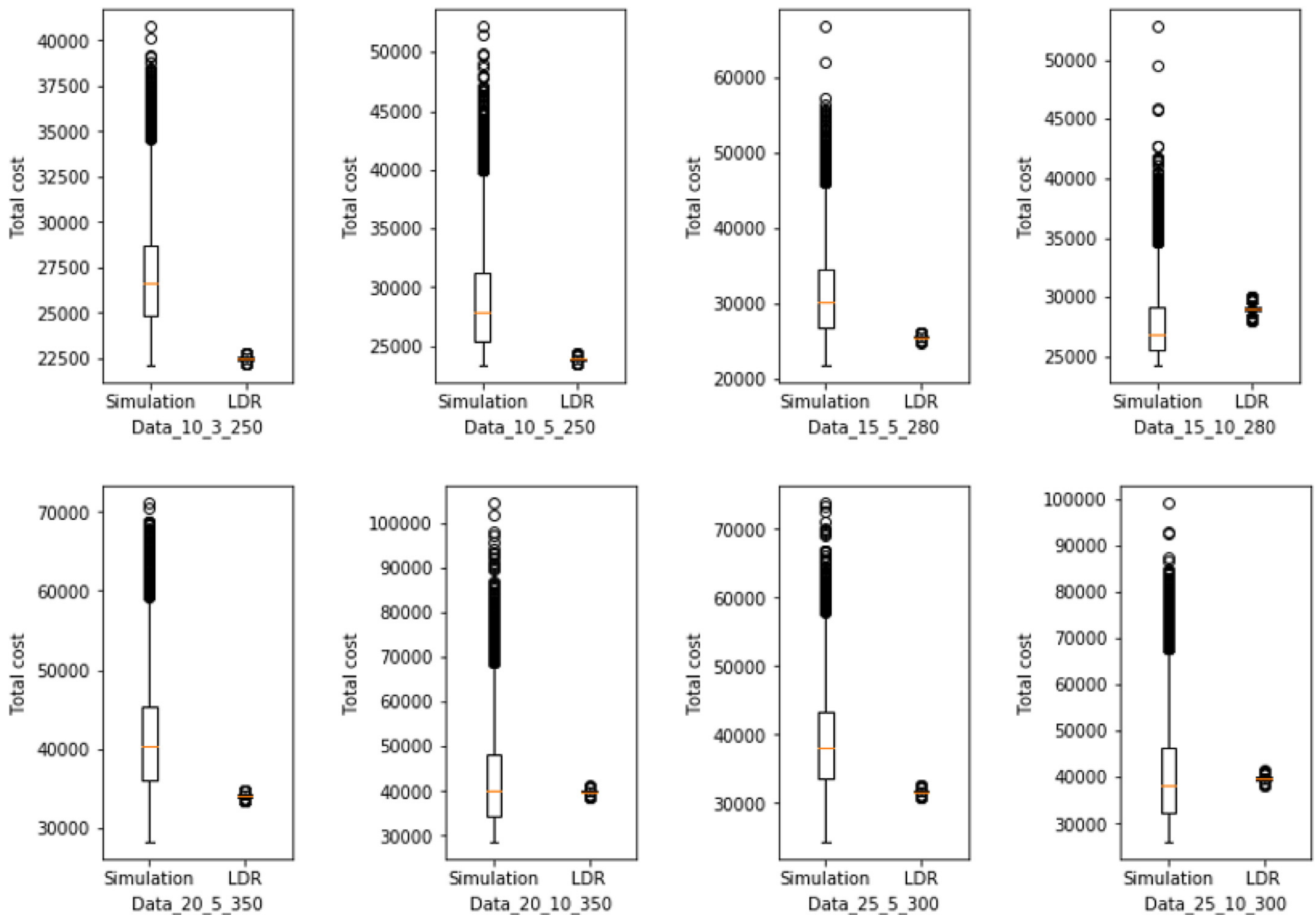

Fig. 6. Results of experiment 2.

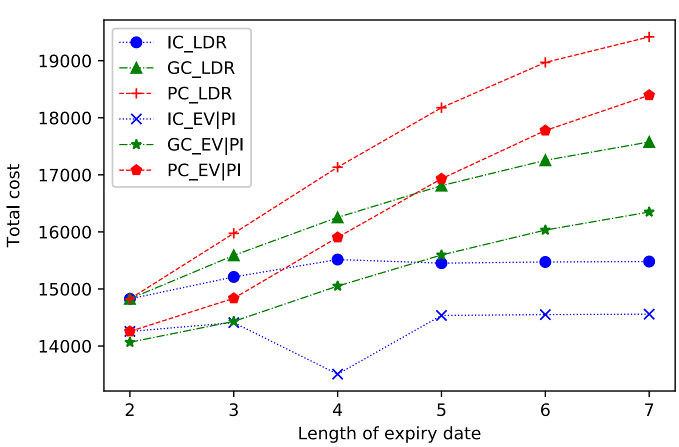

Fig. 7. Comparing between IC, GC, and PC by varying the expiry date τ in Experiment 3.

sets of randomly generated uncertainty factors through the deterministic model (3) and considered the average value as EV|PI. EV|PI was assumed that all information about the distribution of uncertainty factors are known to the inventory manager. Accordingly, EV|PI was used to validate the objective value from the RIMMOR. The results of Experiment 3 are represented in Table 3 and illustrated in Fig. 7. From Table 3 and Fig. 7, we derived the following observations.

Observation 1. In the case of IC, the objective value of the LDR did not show much variability even when τ increased. In contrast, the objective values of GC and PC show an increasing tendency when τ increases.

In the case of IC, sampling data showed that the sum of uncertainty factors converged to 1 near the purchasing date. Although τ increases, most of the revisiting demands near the expiry date were 0. Accordingly, extending the expiry date τ did not have a significant impact and a noticeable difference was not shown in the objective values. For GC, because the revisiting rate was distributed evenly during the period, customers who have the potential to revisit until the expiry date were considered. Accordingly, more conservative solutions were obtained. In the case of PC, the greater the likelihood that the customer revisits in the distant future, the higher the revisiting demand at the end of the planning horizon. Consequently, Experiment 3 showed the results stated in Observation 1.

Observation 2. Results of Experiment 3 show that PC $\geq$ GC $\geq$ IC for the LDR.

In the case of PC, when τ increases, the revisiting demand accumulates at the end of the period. Consequently, more inventory is accumulated in advance to cope with cumulative revisiting de-

Table 3
Results of Experiment 3.

Customer type	Policy	Data (total planning horizon_expiry date_capacity)					
		10_2_280	10_3_280	10_4_280	10_5_280	10_6_280	10_7_280
IC	LDR	14826.8	15211.0	15515.7	15452.3	15473.0	15480.8
	EV|PI	14256.0	14410.4	13507.6	14535.7	14550.8	14558.8
GC	LDR	14827.6	15588.6	16254.3	16810.8	17254.4	17576.3
	EV|PI	14066.3	14431.9	15050.7	15595.7	16032.2	16348.7
PC	LDR	14827.3	15974.8	17133.6	18176.5	18969.5	19417.7
	EV|PI	14256.3	14836.5	15904.1	16931.5	17776.5	18395.5

mands, and a substantial penalty occurs due to the unacceptable quantity of the order. For GC, revisiting demands are spread evenly. As the expiry date τ is extended further from the purchase date, revisiting demand also accumulates at the end of the period. Accordingly, we observed that the objective value increased as τ was extended. Notable results were observed for IC. Generated data showed that the majority took the second item at the purchase date and gradually decreased from the next period. Thus, the backorder cost occurred by the order capacity was less than that of the other two cases and large variability in the objective value was not observed.

Also, as can be seen from Fig. 7, the LDR provided a reasonable upper bound when EV|PI was regarded as a benchmark. Although IC exhibits the tendency related to the gap between LDR and EV|PI when the expiry date τ is 4, this can be interpreted as a smoothing effect of the revisiting demand in EV|PI. To sum up, the results of Experiments 2 and 3 demonstrate that RIMMOR establishes a robust and stable plan against uncertainty factors while providing a reasonable upper bound in the multistage stochastic optimization model.

4.4. Managerial insights

To provide managerial insights, we conducted computational experiments with the three types of customers tendencies. We generated three types of data to incorporate the property of GC, PC, and IC. From the results, we derived the following managerial insights for the retailer:

(i) For certain products, customers are less likely to take both items on the purchasing date and thus have a higher probability of revisiting in the future. For these products, we recommend setting the expiry date not too far in the future. Perishable items or heavy products might be examples of items that are not usually taken at once.

(ii) Products that many customers want to take at the last moment before the expiry date show the greatest cost among the three comparison experiments. This can be the case for a product that is used for a relatively long time. For the retailer, it is necessary to hold a large amount of inventory until the end of the period, but sometimes this is difficult due to the capacity of the order quantity. Therefore, we suggest developing a way to induce customers to make a reservation on the purchase date to pick up the second product on a certain date. Then, customers will not face stockouts and retailers can reduce uncertainty.

(iii) We recommend that BOGO promotions offered through MOR be conducted for daily necessities. In the case of daily necessities, there is a high possibility that customers will return again soon for the second product. Even if the expiry date is set far into the future, the cost during the entire period does not change significantly.

5. Conclusions

MOR is an innovative application that has been downloaded by more than ten million customers. Customers using MOR can revisit the store at a later date to take the second product that they have earned through a BOGO promotion. For retailers, however, it is difficult to respond efficiently with the existing inventory model due to the high level of uncertainty with regard to the revisiting date. Accordingly, we developed the RIMMOR and demonstrated through computational experiments that it could provide a reasonable inventory policy.

The robust optimization approach presented in this study has a distinctive feature that differentiates it from previous studies. The sum of the uncertainty factors in a particular interval is constrained to less than or equal to 1. Constrained uncertainty factors from an inner optimization problem were reformulated into a dual linear program to retain robustness and tractability. The robust counterpart was developed as a second-order cone program, which was tractable until a modest data size. Moreover, the robust counterpart provided a stable solution for the worst case without full information about the distribution. Compared with the EV|PI, which is the substitute for a multistage stochastic optimization model, the robust counterpart provided the reasonable bound derived from only mean, support, and covariance of uncertainty factors.

To the best of our knowledge, this research is the first attempt to develop an IMMOR by adopting the robust optimization approach. Therefore, the opportunity for future research is immense. Naturally, an extension to a multi-item inventory model could be considered. All items could be MOR-based BOGO products or a mix of products that are not part of the promotion. RIMMOR could also be generalized as a buy-x-get-y promotion, or offering other freebies, rather than the BOGO promotion [17]. When the model is generalized, it will give flexibility to the retailer's decision. The promotion which returns the point to the customer, which has a similar mechanism with BOGO can be considered. Moon et al. [20] explored this issue with supply chain coordination. If there is an expiry date of available point and the customer uses the returned point to purchase an additional product, the inventory model presented in this study could be extended. A study on dynamic pricing of BOGO products was conducted by Kim et al. [18]. If MOR is applied, a model simultaneously considering both pricing and inventory can be developed.

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Appendix A

Consider the problem under the planning horizon $t \in \{1, \ldots, 5\}$. Assume an expiry date, τ, as 2 and initial inventory levels, u_1 and v_1, as 0. Table A.1 summarizes the relevant costs of this numerical example. For the capacity of the order quantity, C_t was assumed as 30 for every period. Purchasing demands were assumed as $\mathbf{d}_t = (11, 18, 15, 19)$. Consequently, revisiting demands $\tilde{\xi}_t$ were developed as follows:

$$\tilde{\xi}_1(\tilde{\boldsymbol{\rho}}) = 11\tilde{\rho}_1^1$$
$$\tilde{\xi}_2(\tilde{\boldsymbol{\rho}}) = 11\tilde{\rho}_1^2 + 18\tilde{\rho}_2^2$$
$$\tilde{\xi}_3(\tilde{\boldsymbol{\rho}}) = 18\tilde{\rho}_2^3 + 15\tilde{\rho}_3^3$$
$$\tilde{\xi}_4(\tilde{\boldsymbol{\rho}}) = 15\tilde{\rho}_3^4 + 19\tilde{\rho}_4^4$$

We made the decision variables based on the LDR as follows:

$$\mathbf{x_t}(\tilde{\boldsymbol{\rho}}) = \{x_1^0, x_2^0 + x_2^{1,1}\tilde{\rho}_1^1, x_3^0 + x_3^{1,1}\tilde{\rho}_1^1 + x_3^{1,2}\tilde{\rho}_1^2 + x_3^{2,2}\tilde{\rho}_2^2, x_4^0 + x_4^{1,1}\tilde{\rho}_1^1 + x_4^{1,2}\tilde{\rho}_1^2 + x_4^{2,2}\tilde{\rho}_2^2 + x_4^{2,3}\tilde{\rho}_2^3 + x_4^{3,3}\tilde{\rho}_3^3\}$$
$$\mathbf{y_t}(\tilde{\boldsymbol{\rho}}) = \{y_1^0, y_2^0 + y_2^{1,1}\tilde{\rho}_1^1, y_3^0 + y_3^{1,1}\tilde{\rho}_1^1 + y_3^{1,2}\tilde{\rho}_1^2 + y_3^{2,2}\tilde{\rho}_2^2, y_4^0 + y_4^{1,1}\tilde{\rho}_1^1 + y_4^{1,2}\tilde{\rho}_1^2 + y_4^{2,2}\tilde{\rho}_2^2 + y_4^{2,3}\tilde{\rho}_2^3 + y_4^{3,3}\tilde{\rho}_3^3\}$$

Accordingly, decision variables for the inventory levels also take the affine function of uncertainty factors as follows:

$$\mathbf{u_t}(\tilde{\boldsymbol{\rho}}) = \{u_1^0, u_2^0 + u_2^{1,1}\tilde{\rho}_1^1, u_3^0 + u_3^{1,1}\tilde{\rho}_1^1 + u_3^{1,2}\tilde{\rho}_1^2 + u_3^{2,2}\tilde{\rho}_2^2, u_4^0 + u_4^{1,1}\tilde{\rho}_1^1 + u_4^{1,2}\tilde{\rho}_1^2 + u_4^{2,2}\tilde{\rho}_2^2 + u_4^{2,3}\tilde{\rho}_2^3 + u_4^{3,3}\tilde{\rho}_3^3, u_5^0 + u_5^{1,1}\tilde{\rho}_1^1 + u_5^{1,2}\tilde{\rho}_1^2 + u_5^{2,2}\tilde{\rho}_2^2 + u_5^{2,3}\tilde{\rho}_2^3 + u_5^{3,3}\tilde{\rho}_3^3 + u_5^{3,4}\tilde{\rho}_3^4 + u_5^{4,4}\tilde{\rho}_4^4\}$$

$$\mathbf{v_t}(\tilde{\boldsymbol{\rho}}) = \{v_1^0, v_2^0 + v_2^{1,1}\tilde{\rho}_1^1, v_3^0 + v_3^{1,1}\tilde{\rho}_1^1 + v_3^{1,2}\tilde{\rho}_1^2 + v_3^{2,2}\tilde{\rho}_2^2, v_4^0 + v_4^{1,1}\tilde{\rho}_1^1 + v_4^{1,2}\tilde{\rho}_1^2 + v_4^{2,2}\tilde{\rho}_2^2 + v_4^{2,3}\tilde{\rho}_2^3 + v_4^{3,3}\tilde{\rho}_3^3, v_5^0 + v_5^{1,1}\tilde{\rho}_1^1 + v_5^{1,2}\tilde{\rho}_1^2 + v_5^{2,2}\tilde{\rho}_2^2 + v_5^{2,3}\tilde{\rho}_2^3 + v_5^{3,3}\tilde{\rho}_3^3 + v_5^{3,4}\tilde{\rho}_3^4 + v_5^{4,4}\tilde{\rho}_4^4\}$$

We will derive the balance equations concerning **y, v**, and $\tilde{\boldsymbol{\xi}}$ while omitting the balance equations for **x, u**, and **d** which are easy to show. Balance equations for **y, v**, and $\tilde{\boldsymbol{\xi}}$ for the entire periods $t \in \{1, \ldots, 5\}$ are derived in (A.1) as follows:

$$\begin{aligned}
&v_2^0 + v_2^{1,1}\tilde{\rho}_1^1 = v_1^0 + y_1^0 - 11\tilde{\rho}_1^1 \\
&\Leftrightarrow \begin{cases} v_2^0 = v_1^0 + y_1^0 \\ v_2^{1,1} = -11 \end{cases} \\
&v_3^0 + v_3^{1,1}\tilde{\rho}_1^1 + v_3^{1,2}\tilde{\rho}_1^2 + v_3^{2,2}\tilde{\rho}_2^2 = v_2^0 + v_2^{1,1}\tilde{\rho}_1^1 + y_2^0 + y_2^{1,1}\tilde{\rho}_1^1 - 11\tilde{\rho}_1^2 - 18\tilde{\rho}_2^2 \\
&\Leftrightarrow \begin{cases} v_3^0 = v_2^0 + y_2^0 \\ v_3^{1,1} = v_2^{1,1} + y_2^{1,1} \\ v_3^{1,2} = -11 \\ v_3^{2,2} = -18 \end{cases} \\
&v_4^0 + v_4^{1,1}\tilde{\rho}_1^1 + v_4^{1,2}\tilde{\rho}_1^2 + v_4^{2,2}\tilde{\rho}_2^2 + v_4^{2,3}\tilde{\rho}_2^3 + v_4^{3,3}\tilde{\rho}_3^3 \\
&\quad = v_3^0 + v_3^{1,1}\tilde{\rho}_1^1 + v_3^{1,2}\tilde{\rho}_1^2 + v_3^{2,2}\tilde{\rho}_2^2 + y_3^0 + y_3^{1,1}\tilde{\rho}_1^1 + y_3^{1,2}\tilde{\rho}_1^2 + y_3^{2,2}\tilde{\rho}_2^2 - 18\tilde{\rho}_2^3 - 15\tilde{\rho}_3^3 \\
&\Leftrightarrow \begin{cases} v_4^{1,1} = v_3^{1,1} + y_3^{1,1} \\ v_4^{1,2} = v_3^{1,2} + y_3^{1,2} \\ v_4^{2,2} = v_3^{2,2} + y_3^{2,2} \\ v_4^{2,3} = -18 \\ v_4^{3,3} = -15 \end{cases} \\
&v_5^0 + v_5^{1,1}\tilde{\rho}_1^1 + v_5^{1,2}\tilde{\rho}_1^2 + v_5^{2,2}\tilde{\rho}_2^2 + v_5^{2,3}\tilde{\rho}_2^3 + v_5^{3,3}\tilde{\rho}_3^3 + v_5^{3,4}\tilde{\rho}_3^4 + v_5^{4,4}\tilde{\rho}_4^4 \\
&\quad = v_4^0 + v_4^{1,1}\tilde{\rho}_1^1 + v_4^{1,2}\tilde{\rho}_1^2 + v_4^{2,2}\tilde{\rho}_2^2 + v_4^{2,3}\tilde{\rho}_2^3 + v_4^{3,3}\tilde{\rho}_3^3 + y_4^0 + y_4^{1,1}\tilde{\rho}_1^1 + y_4^{1,2}\tilde{\rho}_1^2 + y_4^{2,2}\tilde{\rho}_2^2 + y_4^{2,3}\tilde{\rho}_2^3 + y_4^{3,3}\tilde{\rho}_3^3 - 15\tilde{\rho}_3^4 - 19\tilde{\rho}_4^4 \\
&\Leftrightarrow \begin{cases} v_5^0 = v_4^0 + y_4^0 \\ v_5^{1,1} = v_4^{1,1} + y_4^{1,1} \\ v_5^{1,2} = v_4^{1,2} + y_4^{1,2} \\ v_5^{2,2} = v_4^{2,2} + y_4^{2,2} \\ v_5^{2,3} = v_4^{2,3} + y_4^{2,3} \\ v_5^{3,3} = v_4^{3,3} + y_4^{3,3} \\ v_5^{3,4} = -15 \\ v_5^{4,4} = -19 \end{cases}
\end{aligned} \tag{A.1}$$

Based on the (A.1), the solution, which is the inventory policy, could be obtained by solving the robust counterpart as follows:

$$\mathbf{x}_t(\tilde{\boldsymbol{\rho}}) = \{16.12, 14.88, 15, 11\}$$
$$\mathbf{y}_t(\tilde{\boldsymbol{\rho}}) = \{13.88, 15.12, 9.88 + 5.12\tilde{\rho}_1^1 + 5.12\tilde{\rho}_1^2, 3.73 + 4.13\tilde{\rho}_1^1 + 4.13\tilde{\rho}_1^2 + 11.14\tilde{\rho}_2^2 + 11.14\tilde{\rho}_2^3\}$$

Table A.1
Related costs of the numerical example.

Cost per unit	Planning horizon t			
	1	2	3	4
c_t	10	10	10	10
h_t	1.45	1.67	1.90	1.57
b_t	4.78	4.92	4.17	4.35
p_t	47.78	49.18	41.65	43.45

Appendix B

We generated a random value from a uniform distribution with support [0, 1] for all non-zero $\tilde{\boldsymbol{\rho}}$. Afterward, all $\tilde{\boldsymbol{\rho}}$ were normalized so that the sum in the same row became 1. For the last elements in each row, we forced the sum of the revisiting rate to be 1. The pseudocode is described in Algorithm 1.

Algorithm 1 Generation of uncertainty factors.

while $i, j \in \mathfrak{T}^-$ **do**
  if $j \geq i$ **then**
    $\tilde{\rho}_i^j \leftarrow$ uniform (0, 1)
  end
end
while $i, j \in \mathfrak{T}^-$ **do**
  $\tilde{\rho}_i^j \leftarrow \tilde{\rho}_i^j / \sum_{j \in \mathfrak{T}^-} \tilde{\rho}_i^j$
end

Appendix C

For IC, we generated the random value based on the uniform distribution whose support is [0, 1]. In succession, the next generated random value has the support between zero and the value obtained by subtracting the cumulative sum of generated random values from 1. We proceeded recursively in this manner and forced the sum from purchasing date to the last revisiting date to be 1. In the case of data generation for PC, IC data was generated and rearranged in the reverse order at the same row. The pseudocodes of IC and PC are described in Algorithms 2 and 3, respectively.

Algorithm 2 Generation of uncertainty factors IC.

while $i, j \in \mathfrak{T}^-$ **do**
  if $j \geq i$ **then**
    $\tilde{\rho}_i^j \leftarrow$ uniform $(0, 1 - \sum_{k=1}^{j-1} \tilde{\rho}_i^j)$
  end
end

Algorithm 3 Generation of uncertainty factors PC.

while $i, j \in \mathfrak{T}^-$ **do**
  if $j \geq i$ **then**
    $\tilde{\rho}_i^j \leftarrow$ uniform $(0, 1 - \sum_{k=1}^{j-1} \tilde{\rho}_i^j)$
  end
end
while $i, j \in \mathfrak{T}^-$ **do**
  while $k \in \{1, \cdots, \tau\}$ **do**
    $\tilde{\rho}_i^{i+k-1} \leftarrow \tilde{\rho}_i^{i+\tau-k}$
  end
end

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.omega.2019.102170.

References

[1] Aharon B-T, Boaz G, Shimrit S. Robust multi-echelon multi-period inventory control. Eur J Oper Res 2009;199(3):922–35.
[2] Ang M, Lim YF, Sim M. Robust storage assignment in unit-load warehouses. Manage Sci 2012;58(11):2114–30.
[3] Ben-Tal A, Golany B, Nemirovski A, Vial J-P. Retailer-supplier flexible commitments contracts: arobust optimization approach. Manuf Serv Oper Manage 2005;7(3):248–71.
[4] Ben-Tal A, Goryashko A, Guslitzer E, Nemirovski A. Adjustable robust solutions of uncertain linear programs. Math Program 2004;99(2):351–76.
[5] Ben-Tal A, Nemirovski A. Robust convex optimization. Math Oper Res 1998;23(4):769–805.
[6] Ben-Tal A, Nemirovski A. Robust solutions of linear programming problems contaminated with uncertain data. Math Program 2000;88(3):411–24.
[7] Bertsimas D, Brown DB, Caramanis C. Theory and applications of robust optimization. SIAM Rev 2011;53(3):464–501.
[8] Bertsimas D, Sim M. The price of robustness. Oper Res 2004;52(1):35–53.
[9] Bertsimas D, Thiele A. A robust optimization approach to inventory theory. Oper Res 2006;54(1):150–68.
[10] Cárdenas-Barrón LE, Chung K-J, Treviño-Garza G. Celebrating a century of the economic order quantity model in honor of ford whitman harris. Int J Prod Econ 2014;155:1–7.
[11] Cárdenas-Barrón LE, González-Velarde JL, Treviño-Garza G. A new approach to solve the multi-product multi-period inventory lot sizing with supplier selection problem. Comput. Oper Res 2015;64:225–32.
[12] Cárdenas-Barrón LE, González-Velarde JL, Treviño-Garza G, Garza-Nuñez D. Heuristic algorithm based on reduce and optimize approach for a selective and periodic inventory routing problem in a waste vegetable oil collection environment. Int J Prod Econ 2019;211:44–59.
[13] Chen W, Sim M. Goal-driven optimization. Oper Res 2009;57(2):342–57.
[14] Gabrel V, Murat C, Thiele A. Recent advances in robust optimization: an overview. Eur J Oper Res 2014;235(3):471–83.
[15] Goh J, Sim M. Robust optimization made easy with rome. Oper Res 2011;59(4):973–85.
[16] Gorissen BL, Yanıkoğlu İ, den Hertog D. A practical guide to robust optimization. Omega 2015;53:124–37.
[17] Huang K-L, Kuo C-W, Shih H-J. Advance selling with freebies and limited production capacity. Omega 2017;73:18–28.
[18] Kim K-K, Lee C-G, Park S. Dynamic pricing with bogo promotion in revenue management. Int J Prod Res 2016;54(17):5283–302.
[19] Lee S, Moon I. Robust empty container repositioning considering foldable containers. Eur J Oper Res 2019.
[20] Moon I, Xu J, Feng X, Ruan X. Cooperative sales promotion with a point-sharing policy: advantages and limitations. Omega 2019.
[21] Sarkar B. Mathematical and analytical approach for the management of defective items in a multi-stage production system. J Cleaner Prod 2019;218:896–919.
[22] Sarkar B, Guchhait R, Sarkar M, Cárdenas-Barrón LE. How does an industry manage the optimum cash flow within a smart production system with the carbon footprint and carbon emission under logistics framework? Int J Prod Econ 2019;213:243–57.
[23] See C-T, Sim M. Robust approximation to multiperiod inventory management. Oper Res 2010;58(3):583–94.
[24] Shampanier K, Mazar N, Ariely D. Zero as a special price: the true value of free products. Mark Sci 2007;26(6):742–57.
[25] Soyster AL. Convex programming with set-inclusive constraints and applications to inexact linear programming. Oper Res 1973;21(5):1154–7.
[26] Tsang H-T, Mak H-Y. Robust optimization approach to empty container repositioning in liner shipping. In: Handbook of Ocean Container Transport Logistics. Springer; 2015. p. 209–29.
[27] Ullah M, Sarkar B. Recovery-channel selection in a hybrid manufacturing-remanufacturing production model with RFID and product quality. Int J Prod Econ 2020;219:360–74.
[28] Wei C, Li Y, Cai X. Robust optimal policies of production and inventory with uncertain returns and demand. Int J Prod Econ 2011;134(2):357–67.
[29] Yanıkoğlu İ, Gorissen BL, den Hertog D. A survey of adjustable robust optimization. Eur J Oper Res 2019;277(3):799–813.