TECHNICAL NOTE --

GENERALIZED FORMULA FOR THE PERIODIC GEOMETRIC-GRADIENT SERIES PAYMENT IN A SKIP PAYMENT LOAN WITH ARBITRARY SKIPS

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ABSTRACT

Recently, Formato derived a useful formula of the amount of periodic equal payment in a skip payment loan with arbitrary skips using a second-order finite difference equation. We rederive his formula using simple arithmetics and provide an intuitive explanation of the formula. We also extend his result to the case that periodic payments occur in a geometric-gradient-series.

INTRODUCTION

A direct reduction loan consists of a series of *equal* periodic payments uniformly separated in time. A skip payment loan is a direct reduction loan in which certain payments are skipped completely. See Formato [1] for the detailed description of this type of loan. Recently, Formato [1] derived a useful formula of the amount of periodic level (or equal) payment in a skip payment loan with arbitrary skips using a second-order finite difference equation. We can rederive his formula easily using time value of money and the skip payment scheme. We also provide an intuitive explanation of the derived formulas. As pointed out by Thuesen and Fabrycky [2], the periodic payments may not occur in an equal series. We extend Formato's [1] result to the case that periodic payments occur in a geometric-gradient-series. Note that this is a general case of equal periodic payments since if we set the gradient to 0, this geometric-gradient-series payment scheme reduces to the equal periodic payment scheme. This note, together with Formato's result, might be used as a case study or a teaching material in the engineering economy class.

A SIMPLE DERIVATION OF THE FORMULA FOR EQUAL PAYMENT

We use the same notations as in Formato [1]:

d = level periodic payment amount,
P = principal amount borrowed,
N = number of periods in the loan schedule,
R = periodic interest rate (as a decimal),
S = total number of skip intervals,
L_k = payment number of the last regular payment at the k^{th} skip interval, and
M_k = payment number at which regular payments resume for the k^{th} skip interval.

For a better understanding of a skip payment loan, refer to Figure 1 in Formato [1]. Formato derived the periodic payment d by formulating the skip loan repayment problem as a second-order finite difference equation in the unpaid principal remaining after each payment. However, we can easily derive the periodic payment d using the time value of money and the skip payment scheme as follows:

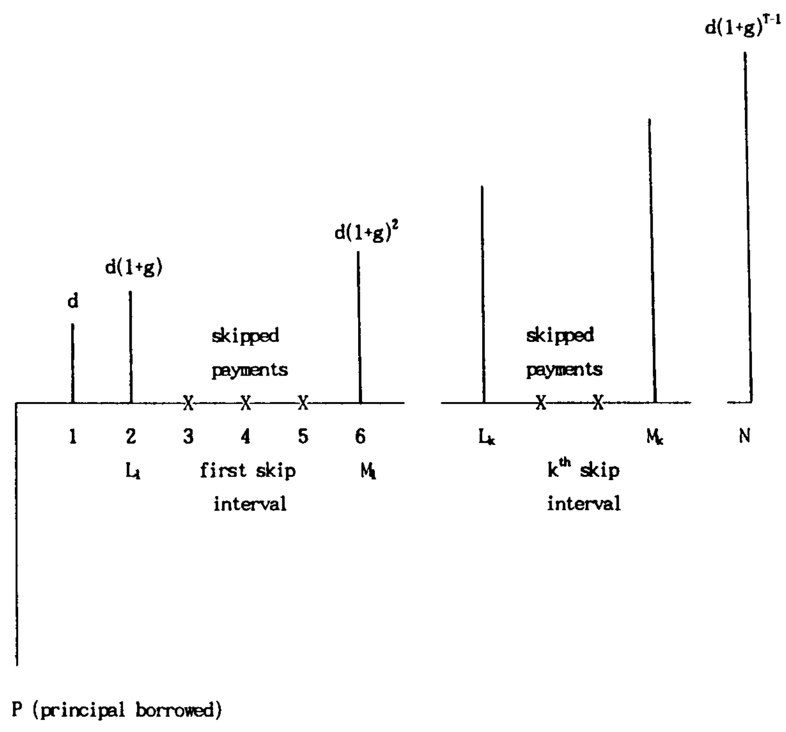

FIGURE 1. Geometric-gradient-series payment in a skip payment direct reduction loan.

$$P = d\left[\sum_{j=1}^{L_1}(1+R)^{-j} + \sum_{j=M_1}^{L_2}(1+R)^{-j} + \ldots + \sum_{j=M_k}^{L_{k+1}}(1+R)^{-j} + \ldots + \sum_{j=M_S}^{N}(1+R)^{-j}\right]$$

$$= \frac{d}{R}\{[1-(1+R)^{-L_1}] + (1+R)^{-M_1+1}[1-(1+R)^{-L_2+M_1-1}] + \ldots +$$

$$(1+R)^{-M_k+1}[1-(1+R)^{-L_{k+1}+M_k-1}] + \ldots +$$

$$(1+R)^{-M_S+1}[1-(1+R)^{-N+M_S-1}]\}$$

$$= \frac{d}{R}\left\{1 + \sum_{k=1}^{S}[(1+R)^{-M_k+1} - (1+R)^{-L_k}] - (1+R)^{-N}\right\} .$$

Consequently, if we solve the above equation for d, we get d as follows which results in the same formula as in Formato [1]:

$$d = \frac{PR(1+R)^N}{\left\{1 + \sum_{k=1}^{S}[(1+R)^{-M_k+1} - (1+R)^{-L_k}]\right\}(1+R)^N - 1} . \quad (1)$$

REMARK 1. Since the skip payment loan is a variation of the direct reduction loan without skips, it is interesting to compare the formulas for the two situations to develop insight into the derived formulas.

If we consider a direct reduction loan with no skips, we get the following equation:

$$P = \frac{d}{R}[1-(1+R)^{-N}] . \quad (2)$$

Now consider a skip payment loan as a combination of a no-skip direct reduction loan and some new borrowings in the future. Whenever the borrower needs to skip a payment, we assume that the borrower can borrow again in the amount d to pay the balance of the no-skip direct reduction loan at that time (See Figure 2). The present value of the k^{th} round of new borrowings is:

$$\frac{d}{R}\left[1-(1+R)^{-(M_k-1)}\right] - \frac{d}{R}\left[1-(1+R)^{-L_k}\right] = \frac{d}{R}\left[(1+R)^{-L_k} - (1+R)^{-M_k+1}\right] .$$

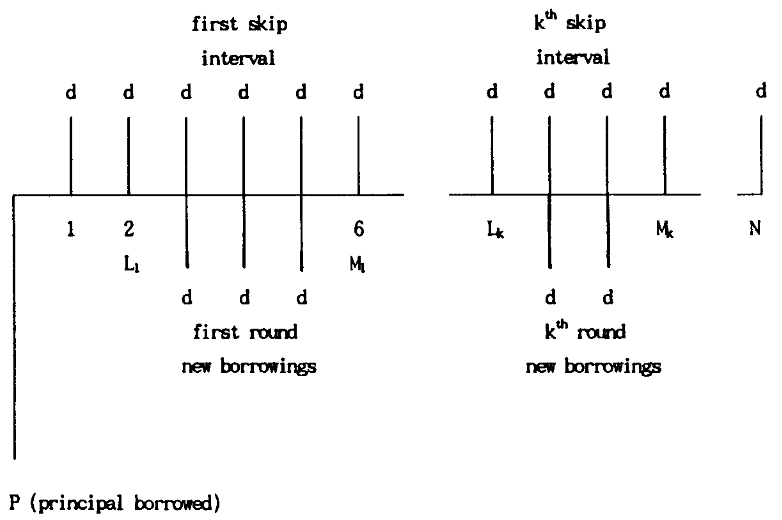

FIGURE 2. Skip payment loan as a combination of a direct reduction loan and some new borrowings.

The present value of all S rounds of new borrowings is therefore

$$\frac{d}{R}\sum_{k=1}^{S}[(1+R)^{-L_k}-(1+R)^{-M_k+1}] \tag{3}$$

The formula for the payment amount in a skip payment loan is derived by subtracting (3) from (2):

$$P=\frac{d}{R}\left\{[1-(1+R)^{-N}]-\sum_{k=1}^{S}[(1+R)^{-L_k}-(1+R)^{-M_k+1}]\right\}$$

The above approach provides an intuitive understanding of the formula. We can see that $[1 - (1 + R)^{-N}]$ corresponds to the direct reduction loan with no skips, and the effect of the skipped payments is accounted for by the terms in $\Sigma^{S}_{k=1}[(1 + R)^{-L_k}\} - (1 + R) - Mk + 1\}]$. We can also deduce that the effect of the skipped payments are weighted bythe timing of the skipped periods via L_k's and M_k's.

FORMULA FOR GEOMETRIC-GRADIENT-SERIES PAYMENT

As pointed out by Thuesen and Fabrycky [2], the periodic payments may not occur in an equal series. In many situations, periodic payments increase or decrease over time by a constant percentage. We extend Formato's result to the geometric-gradient-series payment scheme.

Let d be the amount of first payment, and g be the percentage change (or gradient) in the magnitude of the payment from one period to another. The magnitude of the k^{th} payment is related to payment d as $d(1 + g)^{k-1}$. To better understand the geometric-gradient-series payment in a skip payment loan, refer to Figure 2. Each payment increases by 100 g%. (First payment is d; second payment is $d(1+g)$; payment after first skip is $d(1+g)^{L_1}$; payment after second skip is $d(1+g)^{L_1 + L_2 - M_1 + 1}$; and so on.) If we specify g, the problem reduces to finding the first payment d, which also tells all other payments. Note that this is a general case of the equal periodic payments since if we put the gradient 0, this geometric-gradient-series payment scheme reduces to the equal periodic payment scheme as in the previous Section.

We obtain the following equation using the time value of money and the skip payment scheme depicted in Figure 2:

$$P = d\sum_{j=1}^{L_1}(1+g)^{j-1}(1+R)^{-j} + d\sum_{j=M_1}^{L_2}(1+g)^{L_1-M_1+j}(1+R)^{-j} +$$

$$d\sum_{j=M_2}^{L_3}(1=g)^{L_1+L_2-M_1-M_2+1+j}(1+R)^{-j} + \ldots + d\sum_{j=M_S}^{N}(1+g)^{T-N-1+j}(1+R)^{-j}$$

$$= \left[d(1+g)^{-1}\sum_{j=1}^{L_1} i^j\right] + \left[d(1+g)^{L_1-M_1}\sum_{j=M_1}^{L_2} i^j\right] +$$

$$\left[d(1+g)^{L_1+L_2-M_1-M_2+1}\sum_{j=M_2}^{L_3} i^j\right] + \ldots + \left[d(1+g)^{T-N-1}\sum_{j=M_s}^{N} i^j\right]$$

$$= [d(R-g)^{-1}(1-i^{L_1})] + [d(R-g)^{-1}(1+g)^{L_1-M_1+1}(i^{M_1-1}-i^{L_2})] +$$

$$[d(R-g)^{-1}(1+g)^{L_1+L_2-M_1-M_2+2}(i^{M_2-1}-i^{L_3})] + \ldots +$$

$$[d(R-g)^{-1}(1+g)^{T-N}(i^{M_s-1} - i^{N})] \ .$$

where $T \equiv \Sigma_{k=1}^{S}(Lk - Mk) + S + N$ is total number of periods at which actual payments occur and $i \equiv (1 + g)/(1 + R)$. Solving the above equation for d, we get d as follows:

$$d = \frac{P(R-g)}{X} \tag{4}$$

where

$$X = 1 - i^{L_1} + \sum_{k=1}^{S-1}\left[(i^{M_k-1} - i^{L_{K+1}})(1+g)^{k+\sum_{j=1}^{k}(L_j-M_j)}\right]$$
$$+ [(i^{M_S-1} - i^{N})(1+g)^{T-N}] \ . \tag{5}$$

Note that above formula is valid for $g \neq R$, and the following formula is derived for the special case $g = R$:

$$d = \frac{P}{X}$$

where

$$X = L_1(1+R)^{-1} + \sum_{k=1}^{S-1}\left[(L_{k+1} - M_k + 1)(1+R)^{k-1+\sum_{j=1}^{k}(L_j-M_j)}\right]$$

$$+ [(N - M_S + 1)(1+R)^{T-N-1}].$$

REMARK 2. To see that equation (4) reduces to equation (1) if $g = 0$, substitute $g = 0$ into equation (5) to get:

$$X = 1-(1+R)^{-L_1} + \sum_{k=1}^{S-1}[(1+R)^{1-M_k} - (1+R)^{-L_{k+1}}] + [(1+R)^{1-M_s} - (1+R)^{-N}]$$

$$= \left\{1 + \sum_{k=1}^{S}[(1+R)^{-M_k+1} - (1+R)^{-L_k}] - (1+R)^{-N}\right\}.$$

Consequently, $d = PR/X$ reduces to equation (1).

REMARK 3. We can develop a similar intuitive understanding of the formulas for the geometric-gradient-series payment as in Remark 1 as follows:

If we consider a direct geometric-gradient-series reduction loan, we get the following equation [2]:

$$P = \frac{d}{1+g}\sum_{j=1}^{N} i^j . \tag{6}$$

Now consider a geometric-gradient-series skip payment loan as a combination of a direct geometric-gradient-series reduction loan without skips and some new borrowings in the future. Whenever the borrower needs to skip a payment, we assume that the borrower can borrow again in the appropriate amount to pay the balance owed on the direct reduction loan at that time. The borrowings, however, are different from those in the regular skip payment loan. There are two types of borrowings:

- Type I (k^{th} round new borrowing) occurs during the skip payment periods.
- Type II (k^{th} round new borrowing) occurs during the actual payment periods due to the difference between the amount of payment in a direct geometric-gradient-series reduction loan and that in a geometic-gradient-series skip payment loan (See Figure 3).

The present value of all S rounds new borrowings (Type I) is

$$\frac{d}{1+g}\sum_{k=1}^{S}\sum_{j=L_k+1}^{M_k-1} i^j \tag{7}$$

The present value of all S rounds new borrowings (Type II) is

$$\sum_{k=1}^{S}[d(1+g)^{-1} - d(1+g)^{L_1+\cdots+L_k-M_1-\cdots-M_k+k-1}]\sum_{j=M_k}^{L_{k+1}} i^j \tag{8}$$

where $L_{S+1} \equiv N$.

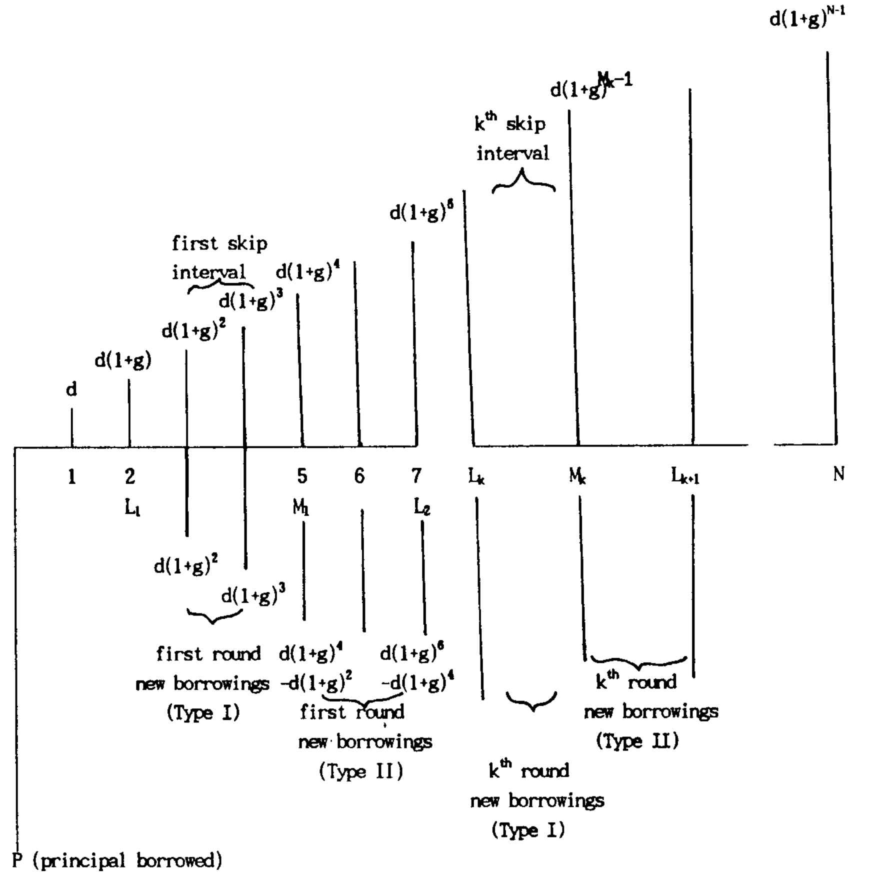

FIGURE 3. Geometric-gradient-series skip payment loan as a combination of a direct geometric-gradient-series reduction loan and some new borrowings.

We can derive the formula for the payment amount in a geometric-gradient-series skip payment loan by subtracting (7) and (8) from (6). Note that equation (6) is analogous to equation (2), and equations (7) and (8) are analogous to equation (3).

The above approach provides an intuitive understanding of the formula. Even though the arithemetic to derive the formula is a little bit tedious, the above formula can be easily coded for implementation on a computer.

A NUMERICAL EXAMPLE

We use the same example in Formato [1]. The data are as follows:

P = \$100,000, $N = 48$, $R = 0.01$, $S = 3$, $L = (8,21,34)$, and $M = (17,28,39)$. In addition, the accountant at Gizmo company suggests a periodic geometric-gradient-series payment scheme with the gradient $g = 0.02$.

Substituting these values into equations (4) and (5) yields a first payment d = \$3,241.70. We can determine all other payments from d. For example, the company will pay \$3,241.70$(1+0.02)^8$ = \$3,798.17 at period 17 (period at which regular payments resume for the 1st skip interval).

ACKNOWLEDGEMENTS

The author is grateful to the helpful suggestions made by two anonymous referees on earlier versions of the paper. The explanations on the insight of the direct reduction loan (Remark 1) is due to a referee.

REFERENCES

[1] Formato, R. A. "Generalized Formula for the Periodic Payment in a Skip Payment Loan with Arbitrary Skips," *The Engineering Economist*, Vol. 37, No. 4 (Summer, 1992), pp. 355-359.

[2] Thuesen, G. J. and W. J. Fabrycky, *Engineering Economy*, 8th edition, Englewood Cliffs, New Jersey, Prentice-Hall Inc., 1993

BIOGRAPHICAL SKETCH

ILKYEONG MOON is an Assistant Professor of Industrial Engineering at Pusan National University in Korea. He earned his Ph.D. in Operations Research from Columbia University in 1991. His interests are in the areas of Production Management, Manufacturing Systems Analysis, and Production Economics. He has research papers published or forthcoming in journals such as *International Journal of Production Research*, *Journal of the Operational Research Society*, *Management Science*, *Omega*, *Operations Research*, and *The Engineering Economist*.