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The Distribution Free Continuous Review Inventory System with a Service Level Constraint

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Abstract

The stochastic inventory models require the information on the lead time demand. However, the distributional information of the lead time demand is often limited in practice. We relax the assumption that the cumulative distribution function, say F, of the lead time demand is completely known and merely assume that the first two moments of F are known and finite. The distribution free approach for the inventory model consists of finding the most unfavorable distribution for each decision variable and then minimizing over the decision variable. We apply the distribution free approach to the continuous review inventory system with a service level constraint. We develop an iterative procedure to find the optimal order quantity and reorder level.

Key words : continuous review, service level, distribution free procedure.

1 Introduction

The continuous review (Q, r) model has been an important inventory model both in the Production Management and Operations Research literature and in practice [3,4,5,6,10,13]. Continuous review systems have the desirable characteristics of being fairly robust to errors in cost estimation and relatively esaily incorporating variable leadtimes. Therefore, continuous review systems will continue to be used and may become more prevalent as the cost of review decreases with advancing technology [14].

Although the continuous review inventory system is quite realistic for describing many real systems, managers often have a difficult time determining an exact value for the stockout cost. In many cases, the stockout cost includes intangible components such as loss of goodwill and potential delays to ohter parts of the system. A common substitute for a stockout cost is a service level which is measured as proportion of demands satisfied directly from stock [10]. In this paper, we study a continuous review inventory system with a constraint on the level of service.

The stochastic inventory models, such as continuous review models and periodic review models, require the information on the lead time demand. However, the distributional information of the lead time demand is often limited in practice. Sometimes all that is available is an educated guess of the mean and the variance. There is a tendency to use the normal distribution under these conditions. However, the normal distribution does not provide the best protection against the occurrence of other distributions with the same mean and the same variance. We relax the assumption that the cumulative distribution function, say F, of the lead time demand is completely known and merely assume that the first two moments of F are known and finite. The minmax distribution free approach for the inventory model consists of finding the most unfavorable distribution for each decision variable and then minimizing over the decision variable. We solve the continuous review model with a service level constraint using the minmax distribution free approach. Refer to

Scarf [11], Gallego [1], Gallego and Moon [2], Moon and Gallego [8], and Moon and Yun [9] for the application of the distribution free approaches to the various production/inventory systems.

2 Basic Model

The problem objective invloves minimizing the average annual ordering cost and inventory carrying costs subject to a constraint on the level of service. Service is measured here as the fraction of demand satisfied directly from stock.

The data and decision variables are as follows:

Q = order quantity (decision variable),
r = reorder point (decision variable),
D = average demand per year,
h = inventory carrying cost per item per year,
K = fixed ordering cost per order,
x = demand during the lead time (random variable),
$f(x)$ = density of demand during the leadtime,
$F(x)$ = cumulative distribution of leadtime demand,
$1-\beta$ = proportion of demands which are met from stock, i.e. service level.

The average annual cost can be written as follows:

$$Min\, G^F(Q,r) = \frac{KD}{Q} + h(\frac{Q}{2} + r - \mu)$$

subject to

$$n(r) \le \beta Q \tag{1}$$

where $n(r) = \int_r^\infty (x-r)f(x)dx$ is the expected number of stockouts per cycle.

In this model, the inventory position of an item is reviewed continuously, and the policy is to order a lot size Q when the inventory position (on hand plus on order minus backorder) drops to a reorder point r.

The Lagrangian function of the annual cost function of the above is as follows:

$$L(Q,r,\lambda) = \frac{KD}{Q} + h(\frac{Q}{2} + r - \mu) + \lambda[n(r) - \beta Q]$$

where λ is a Lagrangian multiplier associated with the service constraint. Upon using Leibniz's rule and set

$\partial L(Q,r,\lambda)/\partial Q = 0, \partial L(Q,r,\lambda)/\partial r = 0, \partial L(Q,r,\lambda)/\partial \lambda = 0$, we get the following first order necessary conditions:

$$\frac{\partial L}{\partial Q} = -\frac{KD}{Q^2} + \frac{h}{2} - \lambda\beta = 0 \tag{2}$$

$$\frac{\partial L}{\partial r} = h - \lambda[1 - F(r)] = 0 \tag{3}$$

$$\frac{\partial L}{\partial \lambda} = n(r) - \beta Q = 0 \tag{4}$$

Combining (2), (3), and (4) , we get the following equation:

$$\frac{KD}{Q^2} = \frac{h}{2} - \frac{h}{1-F(r)}\frac{n(r)}{Q}$$

Solving above equation for Q, we obtain

$$\begin{aligned} Q^F &= \frac{hn(r) + \sqrt{h^2n^2(r) + 2KDh[1-F(r)]^2}}{h[1-F(r)]} \\ &= \frac{n(r)}{1-F(r)} + \sqrt{\left[\frac{n(r)}{1-F(r)}\right]^2 + \frac{2KD}{h}} \end{aligned} \tag{5}$$

From (4), we obtain

$$n(r) = \beta Q \tag{6}$$

Using the following iterative algorithm, we can find the optimal reorder point r^F, and optimal order quantity Q^F.

Algorithm

(Step 0) We use $EOQ = \sqrt{\frac{2KD}{h}}$ as the initial estimate for Q. Call this value Q_0.
(Step 1) Use equation (6) with $Q = Q_0$ to find the reorder point r. Call this value r_0.
(Step 2) Use equation (5) with $r = r_0$ to find Q_1.
(Step 3) Repeat (Step 1) and (Step 2) with $Q = Q_1$, etc. Convergence occurs when $Q_i = Q_{i-1}$ or $r_i = r_{i-1}$ for some i.

Example 1. The average demand per year is 200 units, the ordering cost is \$50, the inventory carrying cost per unit per year is \$2, and the service level is set as 0.98. The lead time demand is normally distributed with a mean (μ) of 100 units and standard deviation (σ) of 25 units. We obtain the optimal order quantity and reorder point as follows: Note that we use

$$n(r) = \sigma L'(\frac{r-\mu}{\sigma})$$

where $L'(\mu) = \int_u^\infty (z-\mu)\phi(z)dz$ which is a unit normal linear-loss integrals [6].

(iter. 1) $Q_0 = \sqrt{\frac{2KD}{h}} = 100.$
Since $L'(\frac{r_0-100}{25}) = 0.08, r_0 = 151.$

(iter. 2) Since $F(r_0) = 0.9783$,
$Q_1 = \sqrt{\left(\frac{2}{0.0217}\right)^2 + \frac{2*50*200}{2}} + \frac{2}{0.0217} = 228.$
Since $L'(\frac{r_1-100}{25}) = 0.182, r_1 = 114$.

(iter. 3) Since $F(r_1) = 0.7123$,
$Q_2 = \sqrt{\left(\frac{4.56}{0.2877}\right)^2 + \frac{2*50*200}{2}} + \frac{4.56}{0.2877} = 117.$
Since $L'(\frac{r_2-100}{25}) = 0.094, r_2 = 124$.

(iter. 4) Since $F(r_2) = 0.8315$,
$Q_3 = \sqrt{\left(\frac{2.34}{0.1685}\right)^2 + \frac{2*50*200}{2}} + \frac{2.34}{0.1685} = 115.$
Since $L'(\frac{r_3-100}{25}) = 0.092, r_3 = 124$.

Using the above algorithm, we get $(Q^N, r^N) = (115, 124)$, and average annual cost of using (Q^N, r^N) is \$251 where $N \in \mathcal{F}$ represents the normal distribution. ∎

3 Distribution Free Model

Now we consider the distribution free approach. We make no assumption on the distribution F of x other than saying that it belongs to the class $\mathcal{F}$ of cumulative distribution functions with mean μ and variance σ^2. If we replace $n(r)$ in equation (1) by its worst case upper bound, we can keep the service level against the worst possible distribution in $\mathcal{F}$. To this end, we need the following proposition.

Proposition 1.

$$E[x-r]^+ \le \frac{\sqrt{\sigma^2+(r-\mu)^2}-(r-\mu)}{2} \quad (7)$$

Moreover, the upper bound (7) is tight. That is, for every r, there exists a distribution $F^* \in \mathcal{F}$ where the bound (7) is tight.

Proof. Notice that

$$(x-r)^+ = \frac{|x-r|+(x-r)}{2}.$$

If we take expectations on the above and use the following Cauchy-Schwarz inequality, we can obtain (7).

$$E|x-r| \le \sqrt{E(x-r)^2} = \sqrt{\sigma^2+(r-\mu)^2}.$$

Now we prove the tightness of the upper bound. For every r, consider the two point cumulative distribution F^* assigning weight

$$\alpha = \frac{\sqrt{\sigma^2+(r-\mu)^2}+(r-\mu)}{2\sqrt{\sigma^2+(r-\mu)^2}}$$

to

$$\mu - \sigma\sqrt{\frac{1-\alpha}{\alpha}} = r - \sqrt{\sigma^2+(r-\mu)^2}$$

and weight

$$1-\alpha = \frac{\sqrt{\sigma^2+(r-\mu)^2}-(r-\mu)}{2\sqrt{\sigma^2+(r-\mu)^2}}$$

to

$$\mu + \sigma\sqrt{\frac{\alpha}{1-\alpha}} = r + \sqrt{\sigma^2+(r-\mu)^2}.$$

Clearly (7) holds with equality and it is easy to verify that $F^* \in \mathcal{F}$. ∎

Our problem is now to solve the following problem:

$$Min\, G^W(Q,r) = \frac{KD}{Q} + h\left(\frac{Q}{2} + r - \mu\right)$$

subject to

$$\frac{\sqrt{\sigma^2+\Delta^2}-\Delta}{2} \le \beta Q \quad (8)$$

where $\Delta \equiv r - \mu$

The Lagrangian function of the above problem is

$$L(Q,\Delta,\lambda) = \frac{KD}{Q} + h\left(\frac{Q}{2}+\Delta\right) + \lambda\left(\frac{\sqrt{\sigma^2+\Delta^2}-\Delta}{2} - \beta Q\right)$$

where λ is a Lagrangian multiplier associated with the service constraint. Upon using Leibniz's rule and set $\partial L(Q,\Delta,\lambda)/\partial Q = 0, \partial L(Q,\Delta,\lambda)/\partial r = 0, \partial L(Q,\Delta,\lambda)/\partial\lambda = 0$, we get the following first order necessary conditions:

$$\frac{\partial L}{\partial Q} = -\frac{KD}{Q^2} + \frac{h}{2} - \lambda\beta = 0 \quad (9)$$

$$\frac{\partial L}{\partial \Delta} = h + \frac{\lambda}{2}\left(\frac{\Delta}{\sqrt{\sigma^2+\Delta^2}} - 1\right) = 0 \quad (10)$$

$$\frac{\partial L}{\partial \lambda} = \frac{\sqrt{\sigma^2+\Delta^2}-\Delta}{2} - \beta Q = 0 \quad (11)$$

Combining (9), (10), and (11), we get the following equation:

$$\frac{KD}{Q^2} = \frac{h}{2} - \frac{h}{Q}\sqrt{\delta^2+\sigma^2}$$

Solving above equation for Q, we obtain

$$Q^W = \sqrt{\sigma^2+\Delta^2} + \sqrt{\sigma^2+\Delta^2+\frac{2KD}{h}} \quad (12)$$

From (11), we obtain

$$\Delta = \frac{\sigma^2}{4\beta Q} - \beta Q \quad (13)$$

Using the following iterative algorithm, we can find the optimal reorder point r^W, and optimal order quantity Q^W. The convergence of the following algorithm can be proven by adopting a similar technique used in Yano [14], and can be found in Moon and Choi [7].

Algorithm

(Step 0) Start from $EOQ = \sqrt{\frac{2KD}{h}} = Q_0$.

(Step 1) Use equation (13) with $Q = Q_0$ to find Δ. Call this value Δ_0.

(Step 2) Use equation (12) with $\Delta = \Delta_0$ to find Q_1.

(Step 3) Repeat (Step 1) and (Step 2) with $Q = Q_1$, etc. Convergence occurs when $Q_i = Q_{i-1}$ or $\Delta_i = \Delta_{i-1}$ for some i. The optimal reorder point is $r = \Delta_i + \mu$.

Example 2. We continue Example 1. From the above algorithm, we can find the optimal order quantity and reorder point, (Q^W, r^W) against the worst distribution. We compare the procedure for the worst case distribution with that for the normal distribution. The results are $(Q^W, r^W) = (164, 145)$ and $(Q^N, r^N) = (115, 124)$ where $N \in \mathcal{F}$ represents the normal distribution. The annual average cost using (Q^W, r^W) is $315 which is about 25% increase compared with using (Q^N, r^N) for the normal distribution. However, if we use (Q^N, r^N) for the worst case distribution, the expected number of shortages per cycle is 5.38. Consequently, the service level results in 0.953 which is much smaller than the prescribed service level 0.98. In other words, we pay additional $64 to keep up with the service level against the worst distribution ∎

4 Concluding Remarks

As pointed by Silver [12], if the quantitative models are to be more useful as aids for managerial decision making, they must represent more realistic problem formulations. In view of this, we have relaxed the assumption that the specific distribution of the lead time demand is known in the continuous review inventory model. We have applied the distribution free approach, and developed a simple iterative algorithm. Further empirical investigations on the robustness of the distribution free approach might be an interesting research problem. We hope this paper will help disseminate the distribution free approach which was originally suggested by Scarf [11] for the newsboy problem.

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