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ORIGINAL RESEARCH

Dynamic versus static rebates: an investigation on price, displayed stock level, and rebate-induced demand using a hybrid bat algorithm

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Abstract
Joint determination of price, rebate, investment in preservation technology, and order quantity is a complex task for retailers today. To help retailers, this paper presents an investigation on a replenishment policy for deteriorating products that focused on the choice between dynamic and static rebates under the price, displayed stock level, and rebate-induced demand. With the objective of maximizing the retailer's profit, six different models were formulated under static and dynamic environments to identify optimal price-and-rebate pair and preservation technology investment policy. Optimal control theory was employed to determine the rate of dynamic rebate. A hybrid bat algorithm (HBA) is developed to find solutions for the proposed non-linear optimization problems. The efficiency of the proposed algorithm was verified with standard test functions. Price sensitivity, the nature of the product, and display stock elasticity were found to be decisive parameters for a retailer's rebate strategy. Dynamic rebate on initial price of the product can significantly improve the profit of the retailer. The retailer's investment decision was also significantly influenced by the nature of the product. Sensitivity analyses were carried out to offer managerial insights.

Keywords Rebate · Inventory · Optimal control · Hybrid bat algorithm

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1 Introduction

Rebates are an integrated part of retailing used to accomplish objectives such as increasing traffic of the retail store by attracting more price-sensitive consumers, accelerating flow of perishable products, eliminating out-of-season inventory. Consequently, the customer rebate has become a prevalent promotional tool for many retailers. In a broad sense, customer rebate promotions include any type of program that involves a partial refund from retailers to consumers upon the purchase of a product without additional conditions. Since the introduction of price and rebate-induced demand by Khouja (2006), researchers have been exploring and analyzing the impacts of various promotional tools offered by the retailer to the consumer. In a common practice, diverse businesses make use of rebates. The retailers of groceries, small and large appliances, convenience items, health and beauty aids, household products, automobiles, liquor, consumer electronics, and computers provide rebates. The amount of the rebate offered depends on factors such as the base retail price, the nature of the product, on-hand inventory, and the replenishment planning horizon. In a study by the Aberdeen Group (2011), approximately 50% of retailers used a frequent rebate program as a part of their promotional mix. These studies have revealed that retailers frequently offer rebates that range between 10 and 70% on high-value products to push the flow of premium products first and thereby move consumer goods. Therefore, appropriate mix of price and rebate becomes an important factor in the business success of modern retailers. With the rapid advancement of information technologies, adjustment of retail price is straightforward.

For this study, we investigated the inventory replenishment decisions of a retailer by considering the effects of static and dynamic rebates on the promotion of deteriorating products under price, display stock level (DSL), and rebate-induced demand. The inventory display is one of the important aspects in retailing to stimulate, educate, and engage customer to buy, and it is one of the key themes of our study. Empirical evidence from both marketing and inventory management literature shows that demand is inventory dependent for apparel retailers (Wolfe 1968), supermarkets (Levin et al. 1972), home improvement stores (Balakrishnan et al. 2008), and magazine retailers (Koschat 2008), among others. In addition, the reduction of the deterioration rate through use of suitable preservation technology is also a challenging issue in retailing. It not only minimizes economic losses but reduces waste, and hence, the environmental impact. The in-depth impacts of rebate and preservation technology investment on the profitability of a monopolistic retailer were explored by considering six different models. We looked at a generalized form of display-inventory-dependent demand. A rebate strategy was also proposed according to the initial retail price. Optimal control theory was used to obtain the analytical form of the dynamic rebate rate. To solve the problems, we hybridized the BA with the Mantegna algorithm, and analyzed the HBA by considering standard non-linear functions. The models were illustrated with some numerical examples, and the effects of the changes in all parameters on the maximum average profit were evaluated through a sensitivity analysis. Our study helps the retailer to select an optimal static or dynamic rebate based on initial price and preservation technology investment strategy to maximize per unit profits.

The organization of the remainder of the paper is as follows. In the following subsection, we present relevant literature and position our research in the literature. Assumptions and background of the models are presented in Sect. 2. Behavior of static and dynamic models are subsequently discussed in Sects. 3 and 4 respectively. The detailed descriptions of the HBA are presented in Sect. 5. Results from comparison and sensitivity analysis are presented in Sect. 6. Finally, conclusions and future research are presented in Sect. 7.

1.1 Literature review

Products such as toys, tools, consumer electronics products, fruits and vegetables, clothing, furniture, books, sports equipment, musical instruments, and pastries are characterized by display-dependent demand features. To explore this aspect of retailing, variability of inventory-level-dependent demand rate on the analysis of the inventory system has been described by several researchers over the past few decades (Urban 1992; Panda et al. 2009; Yang 2014; Xue et al. 2017; Jaggi et al. 2018). An overview of inventory-level-dependent demand can be found in the review article by Urban (2005), who categorized it as additive (linear function of the inventory on hand, Mishra et al. 2017) or multiplicative (non-linear function of the stock level, Saha and Goyal 2015). The major limitation of the linear inventory-level-dependent demand is the condition that more stock on the shelf leads to increased demand. Chang et al. (2006) pointed out this problem and introduced an upper limit for the displayed inventory level on the shelf. They argued that most retail outlets have limited shelf space and "too much piled up in everyone's way leaves a negative impression on buyer and employee alike." In some subsequent studies, such as those from Pal et al. (2005, 2006), a market-oriented three-component demand rate that depends on DSL was proposed. The authors argued that "in competitive market place glamorous display of large number of products with the help of modern light and electric arrangement influences the customers to buy more. But in practice the demand rate would not be dependent on DSL for large stock. It would be DSL dependent within a range and beyond the range it is constant." Gupta et al. (2007) also developed an inventory model of three-component DSL-dependent demand and used an advanced genetic algorithm (GA) to find optimal solutions. Three-component demand is the generalized version of the linear stock-dependent demand problem (Panda et al. 2013; Bhunia et al. 2014). Prasad and Mukherjee (2016) studied an inventory model of perishable items with stock and time-dependent demand in a deterministic environment. They studied the impact of a two-parameter Weibull distribution deterioration rate. Tiwari et al. (2017) explored the replenishment policy for the retailer that leverages stock-dependent demand for non-instantaneous deteriorating items. Particle swarm optimization was used to derive an optimal solution. Mishra et al. (2017) developed an EOQ model with selling price and stock dependent demand for deteriorating products and determined the optimal price, order quantity, and preservation technology investment from the perspective of the retailer. However, to the best of our knowledge, the joint effect of investment in preservation technology and rebate has been ignored in the cited works. Additionally, we consider three-component display stock-dependent demand, which is a more generalized version of a linear-stock-dependent demand.

According to the inventory literature, Ghare and Schrader (1963) first developed a model for exponentially deteriorating inventory and proposed a differential equation governing the variation in the inventory system. Thereafter, researchers and practitioners have been progressively studying inventory systems of deteriorating products and argue in favor of preservation technology investment (PTI) to minimize deterioration rate. Table 1 summarizes some of the key published research on this issue and summarizes the contribution of our study:

From Table 1, one can see that the simultaneous effects of rebate and preservation technology investment have not been studied yet. The retailer consistently tries to accelerate the flow of deteriorating products by introducing several marketing tools that reduce holding and disposal costs, preservation technology investments, and so forth. Therefore, one of the major objectives of this study is to analyze the joint impact of rebate and preservation technology investment on the replenishment decision of a retailer. Table 1 also shows that the efficiency of preservation technology investment is independent of lot size. However, dur-

Table 1 Articles on replenishment decisions under preservation technology investment

Authors	Demand function	Efficiency of PTI	Rebate	Solution technique
Hsu et al. (2010)	Constant	Constant	No	Iterative algorithm
Dye and Hsieh (2012)	Constant	Constant	No	Iterative algorithm
Lee and Dye (2012)	Linear-Stock-dependent	Constant	No	Iterative algorithm
Hsieh and Dye (2013)	Time-varying	Constant	No	Particle swarm optimization
Dye (2013)	Constant	Constant	No	Iterative algorithm
Liu et al. (2015)	Price-quality	Constant	No	Iterative algorithm
Yang et al. (2015a)	Time	Constant	No	Iterative algorithm
Dye and Yang (2016)	Price and reference-price	Constant	No	Mathematica 9
Saha et al. (2017)	Price, reference price and green concern level	Constant	No	Simulated annealing algorithm
Present study	Price - display stock- and rebate	Variable	Dynamic/static	Hybrid bat algorithm

ing retail store operations, centralized temperature control and proper handling of products, including cleaning, sorting, adequate packing, and using appropriate disposal equipment, are crucial. Most retailers rely on these operational improvements to drive additional sales. Therefore, the investment in preservation technology largely depends on order quantity of the retailer, and to formulate models, one must consider the efficiency of preservation technology investment as a function of initial order quantity. Moreover, as a result of the complexity of optimization problems, sometimes researchers are bound to apply metaheuristic algorithms to find replenishment decision solutions under preservation technology investment.

The added complexity of dynamic rebate and efficiency of preservation technology investment inhibits the application of formal methods at the analytical level, which has prompted the use of nature-inspired optimization algorithms. To obtain replenishment decision solutions for real-world complex inventory replenishment problems, nature-inspired optimization algorithms have been found very efficient and have become increasingly popular among researchers. Some of these meta-heuristic algorithms are successfully used in the inventory literature: harmony search (Taleizadeh et al. 2009), ant colony optimization (Nia et al. 2014), simulated annealing (Saha et al. 2017), greedy randomized adaptive search (Bai et al. 2008), particle swarm optimization (Bhunia et al. 2015; Tiwari et al. 2017) and others. In this study, the bat algorithm (BA) is hybridized with Mantegna's algorithm. The resulting HBA has the advantage of simplicity and flexibility. A recent comparison of BA with particle swarm optimization, GA, and other algorithms in the context of e-learning suggested some clear advantages of BA over other algorithms (Gandomi et al. 2013). Mohamed and Moftah (2018) also stated that "the bat algorithm outperforms many other meta-heuristic algorithms, such as PSO and GA, in many optimization fields." A comparative study by considering standard non-linear functions was also conducted to establish the efficiency of the proposed HBA. The proposed algorithm demonstrated faster convergence for our proposed constraint optimization problems.

A market study by Blackhawk Engagement Solutions (2013) demonstrated that across retail categories consumers prefer a high-value rebate over some low-value offers. The problem of making a suitable dynamic pricing strategy for retailers has attracted many researchers. Some recent interesting results about inventory control were published by researchers considering dynamic pricing (Boer 2015; Liu et al. 2015; Dye and Yang 2016; Avinadav et al. 2017; Hu et al. 2017). Differing from cited literature, this paper describes our attempt to verify profitability of retailers using dynamic and static rebates on the basis of initial retail price under displayed stock level (DSL) dependent demand. Because of the great popularity of them, rebates have become an integral part of retailing strategy, and retailers use their own rebate program to attract consumers (Dogan et al. 2010; Yang et al. 2015b). For this study, we explored the way retailers should use dynamic or static rebate to maximize total profits per unit time by determining the optimal rebate strategy under dynamic pricing and the optimal ordering quantity decision under the influence of declining demand in the later stage of a replenishment cycle. To meet this relatively complex objective, we looked at the way the retailer could determine the optimal price-and-rebate pair for the deteriorating items. The pricing scheme applied in this study was similar to a price-skimming. It helps the retailer realize the amount that consumers are willing to pay. If the initial price is too high, the retailer can reduce it easily and sells at a market-clearing price. Moreover, at the later stage, the retailer can attract more price-sensitive consumers. The higher initial price also creates an anchoring effect in consumers' minds as they compare subsequent offers with the higher initial price. For example, despite having limited shelf space compared to that of traditional supermarkets, modern *discount grocery stores* are selling goods by setting high initial prices and then progressively dropping them. Lowering the price of fruit and vegetables at a later stage is a promising day-to-day practice for stimulating the purchase of those products and helps the retailer to get rid of products at the end of the day. In this way, consumers are pleased to buy product at a lower price, which can create an impact on store choice for future shopping (Bell and Lattin 1998; Morley 2017).

2 Notation and assumptions

The following notations are used to develop the models:

2.1 Notation

The study uses the following decision variables:

p	Sales price per unit at time $t \in [0, T]$
Q	Initial inventory level ($Q \geq S_0$)
T	Length of the replenishment cycle
ψ	Preservation technology investment in the replenishment cycle ($\psi \geq 0$)
$R_1(t)$	Dynamic/static rebate rate at any time $t \in [t_2, T]$
$R_2(t)$	Dynamic/static rebate rate at any time $t \in [t_1, t_2]$

The following parameters are used to describe the model:

$I(t)$	Inventory level at any time $t \in [0, T]$
S_c	Replenishment cost per order (\$/order)
c	Purchasing cost per unit (\$/unit)
h	Unit inventory holding cost per unit time (\$/unit/unit time)
c_d	Unit disposal cost per unit (\$/unit)
γ_1	Sensitivity of rebate in demand of the product at time $t \in [t_2, T]$
γ_2	Sensitivity of rebate in demand of the product at time $t \in [t_1, t_2]$
β	Price-elasticity of products, $\beta > \max(\gamma_1, \gamma_2)$
$\theta(\psi)$	Deterioration rate coefficient under preservation technology investment ψ
θ_0	Deterioration rate under natural condition
θ_1	Minimum deterioration rate under preservation technology investment
S_0	Upper limit of DSL dependency parameter
S_1	Lower limit of DSL dependency parameter
t_1	Point of time at which inventory level reaches S_0, $I(t_1) = S_0$
t_2	Point of time at which inventory level reaches S_1, $I(t_2) = S_1$
π^j	j = s1, s2, s3, s4, d1, and d2; denote the retailer's profit in different models

2.2 Assumptions

1. The models take into account the pricing and replenishment decisions of a retailer who sells a deteriorating product and provides a rebate to promote and accelerate the flow of the product. The dynamic rebate rates are determined by optimal control theory. Decision variables are obtained by the HBA algorithm.

2. The market demand rate $D(p, I(t), R_1(t), R_2(t))$ is dependent on the three components of DSL ($I(t)$), sales price (p), and rebate rates ($R_1(t)$ and $R_2(t)$). The functional form of market demand is as follows:

$$D(p, I(t), R_1(t), R_2(t)) = \begin{cases} a + bS_0 - \beta p & \text{if } S_0 \le I(t) \\ a + bI(t) - \beta p + \gamma_2 R_2(t) & \text{if } S_1 \le I(t) \le S_0 \\ a + bS_1 - \beta p + \gamma_1 R_1(t) & \text{if } I(t) \le S_1 \end{cases} \quad (1)$$

We considered the effect of display-stock elasticity in commonly used demand functions, in which demand is linearly decreasing in price and linearly increasing in rebate value (Khouja 2006). If the effects of price and rebate ($\beta = \gamma_1 = \gamma_2 = 0$) can be excluded, then the demand function is similar to that determined by (Bhunia et al. 2014). In the demand function, a (> 0) is the constant demand rate independent of stock level; $b(> 0)$ is the stock-sensitive demand parameter; β represents the sensitivity of demand with respect to retail price; and γ_1 and γ_2 represent the sensitivities of demand with respect to rebates. Inventory is depleted because of the demand rate $a + bS_0 - \beta p$ and deterioration at any time $t \in [0, t_1]$, and it reaches the level of S_0 at time t_1; that is, $I(t_1) = S_0$. Then, inventory is depleted because of demand rate $a + bI(t) - \beta p + \gamma_2 R_2(t)$ and deterioration at any time $t \in [t_1, t_2]$ to time t_2 and reaches the level of S_1, that is, $I(t_2) = S_1$. Finally, the demand becomes independent of stock and the rest of the inventory is depleted to zero in time T; that is, $I(T) = 0$. Thus, the demand rate is a function of DSL in the retail outlet within the range S_0 to S_1. Outside this range, the demand rate becomes constant with respect to the DSL. If $S_0 \to \infty$ and $S_1 \to 0$, then the demand function is converted to linear stack-dependent demand, which is discussed extensively in the literature (Sarkar 2012; Mishra et al. 2017; Liuxina et al. 2018; Jaggi et al. 2018). If $S_1 \to \infty$, then the demand function is converted to two-component stack-dependent demand, which is also discussed extensively in the literature (Panda et al. 2009; Hsieh and Dye 2010; Dye and Hsieh 2011; Tiwari et al. 2017). Therefore, for the present study, we considered a more generalized demand function under inventory-level-dependent demand.

3. The capacity of the retail outlet is greater than S_0. The retailer can allocate sufficient space to keep all the products.

4. The rebate rates are not uniform throughout the replenishment cycle. $R_1(t)$ is the amount of rebate offered by the retailer at any time $t \in [t_2, T]$, $(0 \le R_1(t) \le p - c)$. In a similar fashion, $R_2(t)$ is the amount of rebate offered by the retailer at any time $t \in [t_1, t_2]$, $(0 \le R_2(t) \le p - c)$. Demand rates for products are typically boosted through rebates on selling price. Therefore, studies are needed on the effects of rebate under the three-component DSL-dependent demand. The objective of the retailer is to determine the optimal price-and-rebate pair to maximize profit per unit time. Because of the deteriorating nature of groceries such as fruits, vegetables, meat, and fish, retailers generally suffer from difficulties in maintaining large inventories in a stock-dependent market. In this scenario, retailers offer the necessary amount of DSL and high initial prices at the very beginning of a replenishment cycle. Then, they gradually provide greater rebates such that the higher initial price creates an anchoring effect on price-sensitive consumers.

5. The deterioration rate coefficient $\theta \equiv \theta(\psi) = \theta_1 + (\theta_0 - \theta_1)e^{-\xi\psi Q^{-\lambda}}$ is affected by the investment in preservation technology. According to the literature, the investment in preservation technology is assumed to be independent from order quantity. However, if the initial order quantity (Q) increases, then the retailer needs to invest more. Therefore, the efficiency of the preservation technology is considered as $\xi Q^{-\lambda}$. If $\lambda = 0$, then the investment remains independent from order quantity. In addition, due to configurations and characteristics of the product, the deterioration rate cannot be reduced completely. Therefore, a threshold value of the minimum deterioration rate, θ_1, is considered. If $\theta_1 = 0$ and $\lambda = 0$, then the assumption aligns with that articulated in the literature (Mishra et al. 2017). However, $\theta \to \theta_0$ when $\psi \to 0$; that is, the rate of deterioration remains unchanged if the retailer does

not invest in preservation technology. In a similar pattern, $\theta \to \theta_1$ when $\psi \to \infty$; that is, the rate of deterioration reaches the lowest possible level for a large investment by the retailer in preservation technology.

6. The replenishment rate is instantaneous. Because of the deteriorating nature of products, the inventory level at the end of the replenishment cycle is assumed to be zero. Shortages are not allowed. Similar to Dye (2013), Mishra et al. (2017), Jaggi et al. (2018), Li et al. (2018), Pervin et al. (2018), and Wu et al. (2018), we formulated the model for obtaining an optimal decision for a single replenishment.

3 Static rebate

In this section, four different models for obtaining the replenishment strategies of a retailer are presented. The objective was to determine the optimal retail price, amount of preservation technology investment, order quantity, and amount of rebate to maximize profit per unit time for a retailer selling deteriorating items during a single replenishment cycle. The retailer procures Q units of the product at the beginning of the replenishment cycle. As time passes, the inventory of the retailer decreases due to the combined effects of the market demand and deterioration until time t_1. After that time, the demand decreases as the DSL decreases. The retailer needs to decide the amount of rebate $R_2, t \in [t_1, t_2]$ to stimulate demand, if it is necessary. Finally, after time t_2, demand becomes independent of DSL, and the retailer needs to decide the amount of rebate $R_1, t \in [t_2, T]$ to enhance the flow of product. The depletion of the inventory level throughout the replenishment cycle, $(0, T]$, is governed by the following differential equations:

$$\dot{I}(t) = \begin{cases} -a - bS_0 + \beta p - \theta I(t) & \text{if } S_0 \leq I(t) \\ -a - bI(t) + \beta p - \gamma_2 R_2 - \theta I(t) & \text{if } S_1 \leq I(t) \leq S_0 \\ -a - bS_1 + \beta p - \gamma_1 R_1 - \theta I(t) & \text{if } I(t) \leq S_1 \end{cases} \tag{2}$$

subject to the conditions that $I(0) = Q$, $I(t_1) = S_0$, and $I(t_2) = S_1$. The solutions of Eq. (2) describe the instantaneous inventory level during the replenishment cycle are as follows:

$$I(t) = \begin{cases} Qe^{-\theta t} - (a - \beta p + bS_0)\frac{1-e^{-\theta t}}{\theta} & \text{if } 0 \leq t \leq t_1 \\ S_0 e^{(\theta+b)(t_1-t)} - (a - \beta p + \gamma_2 R_2)\frac{1-e^{(\theta+b)(t_1-t)}}{\theta+b} & \text{if } t_1 \leq t \leq t_2 \\ S_1 e^{\theta(t_2-t)} - (a - \beta p + bS_1 + \gamma_1 R_1)\frac{1-e^{\theta(t_2-t)}}{\theta} & \text{if } t_2 \leq t \leq T \end{cases} \tag{3}$$

Continuity of the inventory levels at t_1 and t_2 and the condition $I(T) = 0$ yield

$$t_1 = \frac{1}{\theta} Log\left[\frac{Q\theta + a + bS_0 - \beta p}{S_0\theta + a + bS_0 - \beta p}\right] \tag{4}$$

$$t_2 = t_1 + \frac{1}{\theta + b} Log\left[\frac{S_0(\theta + b) + a - \beta p + \gamma_2 R_2}{S_1(\theta + b) + a - \beta p + \gamma_2 R_2}\right] \tag{5}$$

$$T = t_2 + \frac{1}{\theta} Log\left[\frac{S_1\theta + a - \beta p + bS_1 + \gamma_1 R_1}{a - \beta p + bS_1 + \gamma_1 R_1}\right] \tag{6}$$

Using the rate of change of inventory in three time intervals, it can be easy to calculate the sells revenue and other system costs which are needed to decide price, investment in preservation technology and how much rebate the retailer can provide:

1. Sells revenue (SR) in the cycle [0, T] can be calculated as follows

$$\begin{aligned} SR &= p\int_0^{t_1}(a+bS_0-\beta p)dt+(p-R_2)\int_{t_1}^{t_2}(a+bI(t)-\beta p+\gamma_2 R_2)dt \\ &\quad +(p-R_1)\int_{t_2}^{T}(a+bS_1-\beta p+\gamma_1 R_1)dt \\ &= p(a+bS_0-\beta p)t_1+(p-R_2)(a-\beta p+\gamma_2 R_2)(t_2-t_1) \\ &\quad +(p-R_1)(a-\beta p+bS_1+\gamma_1 R_1)(T-t_2) \\ &\quad +b(p-R_2)\left(\frac{S_0-S_1}{\theta+b}-\frac{(a-\beta p+\gamma_2 R_2)}{\theta+b}(t_2-t_1)\right) \end{aligned}$$

2. The inventory level is different in three different time intervals, thus the holding cost (HC) and disposal cost (DC) in the cycle [0, T] can be obtained as

$$HC+DC=(h+\theta c_d)\left[\int_0^{t_1}I(t)dt+\int_{t_1}^{t_2}I(t)dt+\int_{t_2}^{T}I(t)dt\right]$$

3. Replenishment cost (RC) and the preservation technology investment (PTI)

$$RC+PTI=S_c+\psi$$

4. To purchase the products, the retailer has to pay

$$PC=cQ$$

Therefore, the total profit of the retailer (TP^{s1}) is given by:

$$\begin{aligned} TP^{s1} &= (b(p-R_2)-(h+\theta c_d))\left(\frac{S_0-S_1}{\theta+b}-\frac{(a-\beta p+\gamma_2 R_2)}{\theta+b}(t_2-t_1)\right) \\ &\quad -\psi-S_c-cQ+p(a+bS_0-\beta p)t_1+(p-R_2)(a-\beta p+\gamma_2 R_2)(t_2-t_1) \\ &\quad +(p-R_1)(a+bS_1-\beta p+\gamma_1 R_1)(T-t_2) \\ &\quad -\frac{(h+\theta c_d)}{\theta}(Q-S_0+S_1-(a+bS_0-\beta p)t_1 \\ &\quad -(a-\beta p+bS_1+\gamma_1 R_1)(T-t_2)) \end{aligned} \tag{7}$$

Therefore, the challenge here is to determine replenishment quantity Q, rebate amounts R_1 and R_2, selling price p, and the preservation technology investment ψ to maximize the total profit per unit time, and one needs to find solution of the following optimization problem to get replenishment decision

$$\text{Maximize} \quad \pi^{s1}(Q,p,\psi,R_1,R_2)=\frac{TP^{s1}}{T} \qquad \text{Prob (1)}$$

subject to,
$t_1=\frac{1}{\theta}Log\left[\frac{Q\theta+a+bS_0-\beta p}{S_0\theta+a+bS_0-\beta p}\right]$, $t_2=t_1+\frac{1}{\theta+b}Log\left[\frac{S_0(\theta+b)+a-\beta p+\gamma_2 R_2}{S_1(\theta+b)+a-\beta p+\gamma_2 R_2}\right]$,
$T=t_2+\frac{1}{\theta}Log\left[\frac{S_1\theta+a-\beta p+bS_1+\gamma_1 R_1}{a-\beta p+bS_1+\gamma_1 R_1}\right]$, $Q\ge S_0$, $p\ge c$, $R_i\ge 0, \theta=\theta_1+(\theta_0-\theta_1)e^{-\xi\psi Q^{-\lambda}}$,
and $p-c\ge R_i$ $\forall i=1,2$.

Differentiating Eq. (6) with respect to R_1 and R_2, one can find that $\frac{\partial T}{\partial R_1}=-S_1\gamma_1/((a+bs_1-p\beta+R_1\gamma_1)(a-p\beta+R_1\gamma_1+S_1(b+\theta)))<0$ and $\frac{\partial T}{\partial R_2}=-(S_0-S_1)\gamma_2/((a-p\beta+R_2\gamma_2+S_0(b+\theta))(a-p\beta+R_2\gamma_2+S_1(b+\theta)))<0$. Therefore, the retailer can reduce replenishment cycle time by providing rebate, and hence accelerate flow of the product and reduce amount of deteriorated product also. The first order condition of optimality with respect to R_1 implies, $\frac{TP^{s1}}{T} = \frac{\partial TP^{s1}}{\partial R_1} / \frac{\partial T}{\partial R_1}$. Therefore, optimal solution exist if $\frac{\partial TP^{s1}}{\partial R_1} = -\frac{1}{\theta^2}\left[\frac{S_1\gamma_1\theta((p-R_1+c_d)\theta+h)}{a-p\beta+R_1\gamma_1+S_1(b+\theta)} + ((a+bS_1-p\beta+(c_d+p-2R_1)\gamma_1)\theta - h\gamma_1)(t_3-t_2)\right] < 0$. The first term of the expression is always positive. Therefore, it will be always decreasing if $\frac{(a+bS_1-(\beta+\gamma_1)p-c_d\gamma_1)\theta-h\gamma_1}{2\gamma_1} > P - R_1$. But, the retailer can earn positive profit if $p - R_1 > c$, and we propose the following proposition:

Proposition 1 *The customer rebate programme is feasible if* $(a+bS_1-(\beta+\gamma_1)p-c_d\gamma_1)\theta - 2c\gamma_1 - h\gamma_1 > 0$. *Customer rebate always reduces replenishment cycle time significantly.*

The above Prob (1) can be extended to verify the profitability of the retailer when the amount of rebate remains uniform. Substituting $R_1 = R_2$ in Eq. (7), the total profit of the retailer (TP^{s2}) can be obtained as follows:

$$\begin{aligned} TP^{s2} = {} & p(a+bS_0-\beta p)t_1 + (b(p-R_1)-(h+\theta c_d)) \\ & \left(\frac{S_0-S_1}{\theta+b} - \frac{(a-\beta p+\gamma_2 R_1)}{\theta+b}(t_2-t_1)\right) \\ & + (p-R_1)(a-\beta p+\gamma_2 R_1)(t_2-t_1) + (p-R_1) \\ & (a-\beta p+bS_1+\gamma_1 R_1)(T-t_2) - cQ - S_c \\ & - \frac{(h+\theta c_d)}{\theta}(Q - S_0 + S_1 \\ & - (a+bS_0-\beta p)t_1 - (a-\beta p+bS_1+\gamma_1 R_1)(T-t_2)) - \psi \end{aligned} \tag{8}$$

Therefore, one needs to find the solution of the the following optimization problem to find replenishment decision where the constraints are obtained by substituting $R_1 = R_2$ in Eqs. (4)–(6).

$$\text{Maximize } \pi^{s2}(Q, p, \psi, R_1) = \frac{TP^{s2}}{T} \qquad \text{Prob (2)}$$

subject to,
$t_1 = \frac{1}{\theta}Log\left[\frac{Q\theta+a+bS_0-\beta p}{S_0\theta+a+bS_0-\beta p}\right]$, $t_2 = t_1 + \frac{1}{\theta+b}Log\left[\frac{S_0(\theta+b)+a-\beta p+\gamma_2 R_1}{S_1(\theta+b)+a-\beta p+\gamma_2 R_1}\right]$, $T = t_2 + \frac{1}{\theta}Log\left[\frac{S_1\theta+a-\beta p+bS_1+\gamma_1 R_1}{a-\beta p+bS_1+\gamma_1 R_1}\right]$

$R_1 \geq 0,\ Q \geq S_0,\ p \geq c, \theta = \theta_1 + (\theta_0 - \theta_1)e^{-\xi\psi Q^{-\lambda}}$, and $p - c \geq R_1$.

Two problems were proposed to verify the profitability of the retailer where a rebate is given as soon as the market demand for the product on display starts to decline. In addition, the solution to an optimization problem is needed to verify profitability of the retailer when the rebate is provided as soon as the demand becomes independent of the DSL; that is, the retailer provides a rebate starting at time t_2. Substituting $R_2 = 0$ in Eq. (7), the total profit of the retailer (TP^{s3}) can be obtained as through the following optimization problem:

$$\begin{aligned} TP^{s3} = {} & (bp-(h+\theta c_d))\left(\frac{S_0-S_1}{\theta+b} - \frac{(a-\beta p)}{\theta+b}(t_2-t_1)\right) - \psi - S_c \\ & + p(a+bS_0-\beta p)t_1 + p(a-\beta p)(t_2-t_1) \\ & + (p-R_1)(a-\beta p+bS_1+\gamma_1 R_1)(T-t_2) - cQ \\ & - \frac{(h+\theta c_d)}{\theta}(Q - S_0 + S_1 \\ & - (a+bS_0-\beta p)t_1 - (a-\beta p+bS_1+\gamma_1 R_1(T-t_2)) \end{aligned} \tag{9}$$

Therefore, one needs to find the solution of the the following optimization problem to get optimal decision where the constraints are obtained by substituting $R_2 = 0$ in Eqs. (4)–(6).

$$\text{Maximize} \quad \pi_s^{SDL}(Q, p, \psi, R_1) = \frac{TP^{s3}}{T} \qquad \text{Prob(3)}$$

subject to, $t_1 = \frac{1}{\theta} Log\left[\frac{Q\theta+a+bS_0-\beta p}{S_0\theta+a+bS_0-\beta p}\right]$, $t_2 = t_1 + \frac{1}{\theta+b} Log\left[\frac{S_0(\theta+b)+a-\beta p}{S_1(\theta+b)+a-\beta p}\right]$, $T = t_2 + \frac{1}{\theta} Log\left[\frac{S_1\theta+a-\beta p+bS_1+\gamma_1 R_1}{a-\beta p+bS_1+\gamma_1 R_1}\right]$

$R_1 \geq 0,\ Q \geq S_0,\ p \geq c, \theta = \theta_1 + (\theta_0 - \theta_1)e^{-\xi\psi Q^{-\lambda}}$, and $p - c \geq R_1$.

Finally, to verify whether the retailer is benefited by providing rebate, one needs to consider the benchmark case where the retailer does not provide any rebate throughout the replenishment cycle. Substituting $R_1 = R_2 = 0$ in Eq. (7), the total profit of the retailer (TP^{s4}) is obtained as follows:

$$TP^{s4} = pbS_0t_1 + p(a - \beta p)T + (bp - (h + \theta c_d))\left(\frac{S_0 - S_1}{\theta + b} - \frac{(a - \beta p)}{\theta + b}(t_2 - t_1)\right) - cQ - S_c$$
$$-\frac{(h + \theta c_d)}{\theta}(Q - S_0 + S_1 - (a + bS_0 - \beta p)t_1 - (a - \beta p + bS_1)(T - t_2)) - \psi \qquad (10)$$

Therefore, one needs to find the solution of the the following optimization problem to get optimal decision where the constraints are obtained by substituting $R_1 = R_2 = 0$ in Eqs. (4)∼(6).

$$\text{Maximize} \quad \pi^{s4}(Q, p, \psi) = \frac{TP^{s4}}{T} \qquad \text{Prob(4)}$$

subject to, $t_1 = \frac{1}{\theta} Log\left[\frac{Q\theta+a+bS_0-\beta p}{S_0\theta+a+bS_0-\beta p}\right]$, $t_2 = t_1 + \frac{1}{\theta+b} Log\left[\frac{S_0(\theta+b)+a-\beta p}{S_1(\theta+b)+a-\beta p}\right]$, $T = t_2 + \frac{1}{\theta} Log\left[\frac{S_1\theta+a-\beta p+bS_1}{a-\beta p+bS_1}\right]$

$Q \geq S_0,\ \theta = \theta_1 + (\theta_0 - \theta_1)e^{-\xi\psi Q^{-\lambda}}$, and $p \geq c$.

The four optimization problems were developed to identify the rebate strategy of the retailer under a static environment. In the next section, expressions for the profit functions and rebate rates of the retailer are under a dynamic environment are presented.

4 Dynamic rebate

In this section two different models for obtaining the rebate rates of a retailer under a dynamic environment are developed. Continuous retail price decreases are often necessary and can boost downward sales to move inventory faster when required. The retailer initially sets a high initial price and drops it depending on the circumstances, particularly for a deteriorating item. This strategy encourages consumers to return to the store in the future also. Similar to the static environment, the retailer procures Q units of product at the beginning of the replenishment cycle and provides a rebate, if necessary, at the rate $R_2(t), t \in [t_1, t_2]$, as soon as the demand and the DSL decrease and at the rate $R_1(t), t \in [t_2, T]$, when the demand becomes independent of the DSL. The challenges include determining simultaneously the optimal initial inventory level, Q; replenishment cycle time, T; selling price, p; rebate rates, $R_1(t), t \in [t_2, T]$ and $R_2(t), t \in [t_1, t_2]$; and preservation technology investment, ψ, to maximize the total profit per unit time of the retailer (π^{d1}). The optimization problem of the retailer under the dynamic environment is as follows:

$$\text{Maximize} \quad \pi^{d1}(Q, p, \psi, R_1(t), R_2(t), T) = \frac{TP^{d1}}{T} \qquad \text{Prob(5)}$$

subject to,

$$\dot{I}(t) = \begin{cases} -a - bS_0 + \beta p - \theta I(t) & \text{if } S_0 \le I(t) \\ -a - bI(t) + \beta p - \gamma_2 R_2(t) - \theta I(t) & \text{if } S_1 \le I(t) \le S_0 \\ -a + \beta p - bS_1 - \gamma_1 R_1(t) - \theta I(t) & \text{if } I(t) \le S_1 \end{cases}$$

$I(0) = Q, I(t_1) = S_0, I(t_2) = S_1, I(T) = 0, t_1 \ge 0, t_2 \ge t_1, T \ge t_2, R_i(t) \ge 0,\ \theta = \theta_1 + (\theta_0 - \theta_1)e^{-\xi\psi Q^{-\lambda}}$, and $p - c \ge R_i(t)$and $i = 1, 2$
where $TP^{d1} = \int_0^{t_1}[p(a - \beta p + bS_0) - (h + \theta c_d)I(t)]dt - cQ - S_c - \psi + \int_{t_1}^{t_2}[(p - R_2(t))(a - \beta p + \gamma_2 R_2(t) + bI(t)) - (h + \theta c_d)I(t)]dt + \int_{t_2}^{T}[(p - R_1(t))(a - \beta p + bS_1 + \gamma_1 R_1(t)) - (h + \theta c_d)I(t)]dt$.

The above optimization problem is a non-linear maximization problem where $R_1(t)$ and $R_2(t)$ represent the control variables, $I(t)$ represents the state variable, and p, Q, ψ, and T represent static variables. By using the conditions $I(0) = Q$ and $I(t_1) = S_0$, the above problem can be restructured as follows:

$$\pi^{d1} = \frac{1}{T}\left[p(a - \beta p + bS_0)t_1 - \frac{(h + \theta c_d)}{\theta}(Q - S_0 - (a + bS_0 - \beta p)t_1) \right. \\ \left. - cQ - S_c - \psi + J_1\right] \qquad \text{Prob(5.1)}$$

subject to,

$$\dot{I}(t) = \begin{cases} -a - bI(t) + \beta p - \gamma_2 R_2(t) - \theta I(t) & \text{if } S_1 \le I(t) \le S_0 \\ -a + \beta p - bS_1 - \gamma_1 R_1(t) - \theta I(t) & \text{if } I(t) \le S_1 \end{cases}$$

$t_1 = \frac{1}{\theta} Log\left[\frac{Q\theta + a + bS_0 - \beta p}{S_0\theta + a + bS_0 - \beta p}\right],\ I(t_1) = S_0,\ I(t_2) = S_1,\ I(T) = 0, t_2 > t_1,\ T > t_2,\ R_i(t) \ge 0,\ p - c \ge R_i(t)\ \forall\, t,\ i = 1, 2.$ where $J_1 = \int_{t_1}^{t_2}[(p - R_2(t))(a - \beta p + \gamma_2 R_2(t) + bI(t)) - (h + \theta c_d)I(t)]dt + \int_{t_2}^{T}[(p - R_1(t))(a - \beta p + bS_1 + \gamma_1 R_1(t)) - (h + \theta c_d)I(t)]dt$.

Now for given ψ, p, and Q, the following optimal control problem is solved first by considering the reduced objective function J_1 to find the control variables as defined below:

$$\text{Maximize} \quad J_1 \qquad \text{Prob(5.2)}$$

subject to,

$$\dot{I}(t) = \begin{cases} -a - bI(t) + \beta p - \gamma_2 R_2(t) - \theta I(t) & \text{if } S_1 \le I(t) \le S_0 \\ -a + \beta p - bS_1 - \gamma_1 R_1(t) - \theta I(t) & \text{if } I(t) \le S_1 \end{cases}$$

$$I(t_1) = S_0,\ I(t_2) = S_1,\ I(T) = 0.$$

As a matter of convenience, the following notations are introduced for simplicity, namely,

$$m_1 = \sqrt{\theta^2 + \theta b}, m_2 = -\sqrt{\theta^2 + \theta b}, M = (h + \theta c_d) - \frac{b(a - \beta p + \gamma_2 p)}{2\gamma_2},$$

$$N = -\frac{(a - \beta p + \gamma_2 p)}{2}$$

$$U = \frac{M(\theta + b/2) - \frac{Nb^2}{2\gamma_2}}{\theta(\theta + b)}, V = \frac{M\gamma_2 - 2N(\theta + b/2)}{2\theta(\theta + b)}, X = \frac{h + \theta c_d}{\theta}$$

$$c_1 = \frac{\frac{b^2(S_1+V)}{2\gamma_2} - (U + \phi e^{-\theta(T-t_2)} - X(1 - e^{-\theta(T-t_2)})(\theta + b/2 - m_2)}{(m_2 - m_1)e^{m_1 t_2}},$$

$$c_2 = \frac{\frac{b^2(S_1+V)}{2\gamma_2} - (U + \phi e^{-\theta(T-t_2)} - X(1 - e^{-\theta(T-t_2)})(\theta + b/2 - m_1)}{(m_1 - m_2)e^{m_2 t_2}},$$

Pontryagin's maximum principle is used to find the dynamic variables for a specific replenishment cycle, preservation technology investment, and price of the product (Sethi and Thompson 2000). If $\lambda(t)$ is the adjoint variable associated with the state equation $\dot{I}(t)$, then the Hamiltonian function (H) for the optimal control problem, depicted by Prob(5.1), can be formulated as follows:

$$H = \begin{cases} (p - R_2(t))(a - \beta p + \gamma_2 R_2(t) + bI(t)) - (h + \theta c_d)I(t) & \text{if } t_1 \le t \le t_2 \\ +\lambda(t)(-a - bI(t) + \beta p - \gamma_2 R_2(t) - \theta I(t)) & \\ (p - R_1(t))(a - \beta p + bS_1 + \gamma_1 R_1(t)) - (h + \theta c_d)I(t) & \text{if } t_2 \le t \le T \\ +\lambda(t)(-a + \beta p - bS_1 - \gamma_1 R_1(t) - \theta I(t)) & . \end{cases} \tag{11}$$

The adjoint variable $\lambda(t)$ must satisfy the following differential equations:

$$\dot{\lambda}(t) = \begin{cases} -\frac{\partial H}{\partial I} = -b(p - R_2(t)) + (h + \theta c_d) + (\theta + b)\lambda(t) & \text{if } t_1 \le t \le t_2 \\ -\frac{\partial H}{\partial I} = (h + \theta c_d) + \theta\lambda(t) & \text{if } t_2 \le t \le T. \end{cases} \tag{12}$$

The first-order conditions for the maximization of the Hamiltonian H with respect to $R_1(t)$ and $R_2(t)$ are obtained by solving $\frac{\partial H}{\partial R_1(t)} = 0$ and $\frac{\partial H}{\partial R_2(t)} = 0$ respectively. On simplification,

$$R_1(t) = \frac{(\gamma_1 + \beta)p - a - \gamma_1\lambda(t) - bS_1}{2\gamma_1}, \ t_2 \le t \le T \tag{13}$$

$$R_2(t) = \frac{(\gamma_2 + \beta)p - a - \gamma_2\lambda(t) - bI(t)}{2\gamma_2}, \ t_1 \le t \le t_2 \tag{14}$$

Moreover, $\frac{\partial^2 H}{\partial R_1(t)^2} = -2\gamma_1 < 0$, $\frac{\partial^2 H}{\partial R_2(t)^2} = -2\gamma_2 < 0$, and $\frac{\partial^2 H}{\partial R_1(t)^2}\frac{\partial^2 H}{\partial R_2(t)^2} - \left(\frac{\partial^2 H}{\partial R_1(t)\partial R_2(t)}\right) = 4\gamma_1\gamma_2 > 0$, therefore H is concave with respect to $R_1(t)$ and $R_2(t)$. Now, backward substitution technique is used to find rebate rates. By using the transversality condition as $\lambda(T) = \phi$, one can obtain the following equation representing adjoint variable

$$\lambda(t) = \phi e^{-\theta(T-t)} - X(1 - e^{-\theta(T-t)}), \ t_2 \le t \le T \tag{15}$$

Using the value of $\lambda(t)$ and the condition $I(t_2) = S_1$, one can obtain the inventory as

$$\begin{aligned} I(t) = & S_1 e^{\theta(t_2-t)} - (a - \beta p + bS_1 + \gamma_1 p)\frac{1 - e^{\theta(t_2-t)}}{2\theta} \quad t_2 \le t \le T \\ & + \frac{\gamma_1}{2}\left[X\left(\frac{e^{-\theta(T-t)}(1 - e^{2\theta(t_2-t)})}{2\theta} - \frac{1 - e^{\theta(t_2-t)}}{\theta}\right) + \phi\frac{e^{-\theta(T-t)}(1 - e^{2\theta(t_2-t)})}{2\theta}\right] \end{aligned} \tag{16}$$

Simplifying the condition $I(T) = 0$, the value of ϕ used for transversality condition is obtained as follows

$$\phi = \frac{2\theta\left[(a-\beta p+bS_1+\gamma p)\frac{1-e^{\theta(t_2-T)}}{\theta} - S_1 e^{\theta(t_2-T)} - \frac{\gamma_1 X}{2}\left(\frac{1-e^{2\theta(t_2-T)}}{2\theta} - \frac{1-e^{\theta(t_2-T)}}{\theta}\right)\right]}{\gamma_1(1-e^{2\theta(t_2-T)})} \tag{17}$$

By eliminating $R_1(t)$ and $R_2(t)$ obtained in Eqs. (13) and (14), one can find that the inventory level $I(t)$ and the adjoint variable $\lambda(t)$ at time $t \in [t_1, t_2]$ are governed by the following differential equations:

$$\left[D - \left(\theta + \frac{b}{2}\right)\right]\lambda_1(t) + \frac{b^2 I(t)}{2\gamma_2} = M \tag{18}$$

$$-\frac{\lambda_1(t)\gamma_2}{2} + \left[D + \left(\theta + \frac{b}{2}\right)\right] I = N \tag{19}$$

where $D \equiv \frac{d}{dt}$. After solving Eqs. (18) and (19), the optimal path representing the inventory level and adjoint variable at any time $t \in [t_1, t_2]$ are obtained by the following equations:

$$I(t) = \frac{2\gamma_2 c_1}{b^2}\left(\theta + \frac{b}{2} - m_1\right)e^{m_1 t} + \frac{2\gamma_2 c_2}{b^2}\left(\theta + \frac{b}{2} - m_2\right)e^{m_2 t} - V \tag{20}$$

$$\lambda_1(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} - U \tag{21}$$

using the continuity conditions $I(t_2) = S_1$ and $\lambda_1(t_2) = \phi e^{-\theta(T-t_2)} - X(1 - e^{-\theta(T-t_2)})$, one may obtain the values of c_1 and c_2 as considered earlier. Moreover, the inventory level must be continuous at the point $t = t_1$, that is,

$$\frac{2\gamma_2 c_1}{b^2}\left(\theta + \frac{b}{2} - m_1\right)e^{m_1 t_1} + \frac{2\gamma_2 c_2}{b^2}\left(\theta + \frac{b}{2} - m_2\right)e^{m_2 t_1} = S_0 + V \tag{22}$$

Therefore, the optimization problem Prob (5) is finally defined as follows:

$$\begin{aligned}\text{Max}\pi^{d1} = \frac{1}{T}\Bigg[&\left(p - c_d - \frac{h}{\theta}\right)(a - \beta p + bS_0)t_1 \\ &- \frac{(h+\theta c_d)}{\theta}(Q - S_0) - cQ - S_c - \psi + J_1\Bigg] \qquad \text{Prob}(5.3)\end{aligned}$$

subject to,

$$I(t) = \begin{cases} Qe^{-\theta t} - (a - \beta p + bs_0)\frac{1-e^{-\theta t}}{\theta} & \text{if } 0 \le t \le t_1 \\ \frac{2\gamma_2}{b^2}\left[c_1(\theta + \frac{b}{2} - m_1)e^{m_1 t} + c_2(\theta + \frac{b}{2} - m_2)e^{m_2 t}\right] - V & \text{if } t_1 \le t \le t_2 \\ S_1 e^{\theta(t_2-t)} - (a - \beta p + \gamma_1 p + bS_1)\frac{1-e^{\theta(t_2-t)}}{2\theta} + & \text{if } t_2 \le t \le T \\ \frac{\gamma_1}{2}\left[X\left(\frac{e^{-\theta(T-t)}(1-e^{2\theta(t_2-t)})}{2\theta} - \frac{(1-e^{\theta(t_2-t)})}{\theta}\right) + \phi\frac{e^{-\theta(T-t)}(1-e^{2\theta(t_2-t)})}{2\theta}\right] & \end{cases}$$

$$\lambda(t) = \begin{cases} c_1 e^{m_1 t} + c_2 e^{m_2 t} - P & \text{if } t_1 \le t \le t_2 \\ \phi e^{-\theta(T-t)} - X(1 - e^{-\theta(T-t)}) & \text{if } t_2 \le t \le T \end{cases}$$

$$R_1(t) = \frac{(\gamma_1+\beta)p - a - \gamma_1\lambda(t) - bS_1}{2\gamma_1},\ t_2 \le t \le T, \quad R_2(t) = \frac{(\gamma_2+\beta)p - a - \gamma_2\lambda(t) - bI(t)}{2\gamma_2},\ t_1 \le t \le t_2$$

$$\frac{2\gamma_2 c_1}{b^2}(\theta + \frac{b}{2} - m_1)e^{m_1 t_1} + \frac{2\gamma_2 c_2}{b^2}(\theta + \frac{b}{2} - m_2)e^{m_2 t_1} = S_0 + V,\ t_1 = \frac{1}{\theta}Log\left[\frac{Q\theta + a + bS_0 - \beta p}{S_0\theta + a + bS_0 - \beta p}\right]$$

$$\theta = \theta_1 + (\theta_0 - \theta_1)e^{-\xi\psi Q^{-\lambda}},\ \psi \ge 0,\ p \ge c + R_2(t_2),\ p \ge c + R_1(t_3),\ T \ge 0.$$

Substituting the value of $\lambda(t)$ from Eq. (15) into Eq. (13), and differentiating partially with respect to t, one can obtain

$$\frac{\partial R_1(t)}{\partial t} = \frac{e^{\theta(t+T)}(((a-bs_1+p\beta-c_d\gamma_1-p\gamma_1)\theta+h\gamma_1)(1-e^{-(T-t_2)\theta})+2s_1\theta e^{-(T-t_2)\theta})}{\gamma_1(e^{2T\theta}-e^{2t_2\theta})}>0$$

Therefore, the rate of rebate increases as time progress. It justifies the reality, the retailer always gives more rebate at the ending of replenishment cycle. Therefore, the following proposition is proposed.

Proposition 2 *The rate of rebate in the final phase increases as time passes.*

Similar to pervious section, if the retailer does not provide any rebate when the demand remains sensitive to the display stock, i.e. $R_2(t)=0,\ \forall\, t\in[t_1,t_2]$, then the profit function per unit time of the retailer converts into

$$\begin{aligned}\pi^{d2} = \frac{1}{T}\Bigg[&\int_0^{t_1}[p(a-\beta p+bS_0)-(h+\theta c_d)I(t)]dt\\ &+\int_{t_1}^{t_2}[p(a-\beta p+bI(t))-(h+\theta c_d)I(t)]\,dt\\ &+\int_{t_2}^{T}[(p-R_1(t))(a-\beta p+bS_1+\gamma_1 R_1(t))\\ &-(h+\theta c_d)I(t)]\,dt-cQ-S_c-\psi\Bigg]\end{aligned}$$

Following the similar way, the optimization problem can be represented as follows:

$$\text{Max}\quad \pi^{d2} \qquad \text{Prob(6)}$$

subject to,

$$I(t)=\begin{cases} Qe^{-\theta t}-(a-\beta p+bs_0)\frac{1-e^{-\theta t}}{\theta} & \text{if } 0\le t\le t_1\\ S_0e^{(\theta+b)(t_1-t)}-(a-\beta p)\frac{1-e^{(\theta+b)(t_1-t)}}{\theta+b} & \text{if } t_1\le t\le t_2\\ S_1e^{\theta(t_2-t)}-(a-\beta p+bS_1\gamma_1 p)\frac{1-e^{\theta(t_2-t)}}{2\theta}+ & \text{if } t_2\le t\le T\\ \frac{\gamma_1}{2}\left[X\left(\frac{e^{-\theta(T-t)}(1-e^{2\theta(t_2-t)})}{2\theta}-\frac{(1-e^{\theta(t_2-t)})}{\theta}\right)+\phi\frac{e^{-\theta(T-t)}(1-e^{2\theta(t_2-t)})}{2\theta}\right] & \end{cases}$$

$R_1(t)=\frac{(\gamma_1+\beta)p-a-\gamma_2\lambda_1}{2\gamma_1},\ t_2\le t\le T,\quad \lambda(t)=\phi e^{-\theta(T-t)}-X(1-e^{-\theta(T-t)}),\ t_2\le t\le T$

$t_1=\frac{1}{\theta}Log\left[\frac{Q\theta+a+bS_0-\beta p}{S_0\theta+a+bS_0-\beta p}\right]$, $t_2=t_1+\frac{1}{\theta+b}Log\left[\frac{S_0(\theta+b)+a-\beta p}{S_1(\theta+b)+a-\beta p}\right]$, $\theta=\theta_1+(\theta_0-\theta_1)e^{-\xi\psi Q^{-\lambda}}$, $\psi\ge 0,\ p\ge c+R_1(t_3),\ T\ge t_2$.

This study proposed only nonlinear constraint optimization problems. It is difficult to find the optimum solution via an analytical approach. In addition, the number of decision variables expanded exponentially in both the objective function and constraints. To obtain the solution for such complex real-world problems, nature-inspired optimization algorithms have been found to be very efficient and have become increasingly popular among researchers. In this study, an HBA is employed and described in Sect. 5.

5 A hybrid bat algorithm

Yang (2010) introduced the BA, which was inspired by the fascinating echolocation behavior of microbats (Colin 2000). Most microbats are insectivores and use sonar to detect prey, avoid obstacles, and locate roosting crevices. In the BA, echolocation behavior of microbats is incorporated into the design of an optimization algorithm. After the pioneer algorithm was introduced, the standard BA and variants have been applied in almost every area of optimization (Yang 2013). In addition, Huang et al. (2013) showed that the BA provides guaranteed global convergence under suitable conditions. The algorithm is based on the following rules:

1. All bats fly randomly with velocity v_i at position x_i. They can automatically adjust frequency of their emitted pulses and adjust the rate of pulse emission $r \in (0, 1)$ on the basis of the proximity of their target.
2. The loudness can vary in many ways; however, the loudness assumedly varies from a large positive number A_0 to a minimum value A_{min}. Also, the frequency value of the ultrasonic burst lies in a range between $[f_{min}, f_{max}]$.

The BA has some limitations that have inspired researchers to propose variations on the original. The orthodox BA reflects a balanced trade-off between particle swarm optimization and an exhaustive local search that is controlled by loudness and pulse rate. An exploration technique was required to decrease the probability of premature convergence and increase the diversity of exploration. Therefore, we introduced Lévy flight to explore the unknown large-scale search space. In addition, the inertia function was introduced to add virtual mass that stabilized the simulated motion of bats and accelerated the convergence rate. The velocity updating equation of the BA is $v_i^{t+1} = v_i^t + (x_i^t - x^*) f_i$. We replaced the x^* with a random y, which is any solution in the population $(f(y) > f(x_i))$ in a maximization problem, to enhance the intensity of the local search within the population. We used two different velocity functions for exploration and exploitation. The velocity equation for the local search is $v_i^{t+1} = v_i^t r + (y - x_i^t) f_i$, where r is generated by using the chaos function. In this process, we changed the direction of the velocity vector so that it can perform an extensive global search. The chaos function was used to produce a chaos number based on the logistics map, which is a one-dimensional map that generates a chaotic sequence: $y_{n+1} = r_0 y_n(1 - y_n)$, $y_0 = 0.19$, and $r_0 = 4$. The first 2000 iterations of the logistic map were discarded to exclude the transient motion that leads to the chaotic attractor. To verify the efficiency of the algorithm, we conducted a comparative study by considering standard non-linear functions, and corresponding results are given in "Appendix A". The detailed pseudo-code of the HBA is as follows:

A hybrid bat algorithm (HBA)

Initialize the bat population x_i and v_i, frequencies f_i, pulse rete r_i, loudness A_i, switch probability p, and parameter for Lévy distribution λ $(i = 1, 2, \cdots, N)$
Define objective function $\pi(x)$, $x \in R^n$. **Set the parameter r by using Chaos-Funtion()**
Find the current global best solution : x^*
While(t < Maximum number of iterations)
  If $p > \varepsilon, \varepsilon \in (0, 1)$
    For each x_i in **N**
      Draw a step vector **L** by using **Mantegna Algorithm**
        Compute $\sigma^2 = \left[\frac{\Gamma(1+\lambda)}{\lambda\Gamma(1+\lambda/2)} \frac{sin(\pi\lambda/2)}{2^{(\lambda-1)/2}}\right]^{1/\lambda}$
        Generate $U \sim (0, \sigma^2)$ and $V \sim (0, 1)$. Compute $L = \frac{U}{|V|^{1/\lambda}}$
      Conduct global exploration :
        $v_i^{t+1} = rv_i^t + rL(x^* - x_i^t)$, $x_i^{t+1} = x_i^t + v_i^{t+1}$
    End For
    If $A_i^t > \varepsilon$ **AND** $\pi(x_i) > \pi(x^*)$, $\varepsilon \in (0, 1)$
      Accept the new solution
      decrease A_i: $A_i^{t+1} = A_i^t * \alpha$
      increase r_i: $r_i^{t+1} = r_i^0(1 - e^{-\gamma t})$
    End If
  End If
  For each x_i in **N**
    Update velocities and solutions using:
    Choose y among the bat population which has better solution than x_i
      $f_i = f_{min} + \varepsilon(f_{max} - f_{min})$, $\varepsilon \in (0, 1)$
      $v_i^{t+1} = rv_i^t + f_i(y - x_i^t)$
      $x_i^{t+1} = x_i^t + v_i^{t+1}$
    If $r_i^t < \varepsilon, \varepsilon \in (0, 1)$
      Generate a local solution around the best solution
        $x_i^{t+1} = x^* + r\varepsilon$, $\varepsilon \in (-A_i^t, A_i^t)$
    End If
    If $A_i^t > \varepsilon$ **AND** $\pi(x_i) > \pi(x^*)$, $\varepsilon \in (0, 1)$
      Accept the new solution
      decrease A_i: $A_i^{t+1} = A_i^t * \alpha$
      increase r_i: $r_i^{t+1} = r_i^0(1 - e^{-\gamma t})$
    End If
  End For
  Find the current global best solution up to t: x^*
  $t = t + 1$
End While
output the best solution x^* and π

In this study, the penalty method (Deb 2000) is used to find optimal decision for the developed non-linear constraint optimization problem. Note that, if the optimization problem is of the following form

Table 2 Optimal price of the retailer, cycle time, amount of rebate, preservation technology investment, order quantity, and profit in the static environment

		p	t_1	t_2	T	R_1	R_2	ψ	Q	Profit
π^{s1}	With PTI	140.13	5.62	9.29	11.32	91.62	46.44	4282.77	757.29	1412.12
	Without PTI	104.44	0.71	3.18	5.36	24.16	–	–	301.39	539.98
π^{s2}	With PTI	138.65	5.78	8.71	11.82	65.21	65.21	4339.51	783.28	1299.82
	Without PTI	94.46	0.51	2.74	5.33	0	0	–	272.86	497.71
π^{s3}	With PTI	114.16	4.57	9.07	11.19	51.37	–	3935.97	708.07	1232.72
	Without PTI	104.44	0.71	3.18	5.36	24.16	–	–	301.39	539.98
π^{s4}	With PTI	100.89	3.89	7.40	11.50	–	–	3685.42	652.63	1007.00
	Without PTI	94.46	0.51	2.74	5.33	–	–	–	272.86	497.71

$$\min f(x),\ x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$$
$$\text{subject to } g_i(x) \le 0,\ (i = 1, 2, \ldots, m)\ h_j(x) = 0\ (j = 1, 2, \ldots, r)$$

then, the pseudo-objective function

$$\min f_1(x) = f(x) + \sum_{i=1}^{m} \mu_i \max\{0, g_i(x)\}^2 + \sum_{j=1}^{r} \phi_j \mid h_j(x) \mid$$

is used to convert it into an unconstrained optimization problem, where μ_i, $(i = 1, 2, \ldots, m)$ and ϕ_j, $(j = 1, 2, \ldots, r)$ are non-negative penalty factors.

6 Computational experiments

In this section, numerical examples, conducted to gain insight into the proposed model, are presented. In addition, a sensitivity analysis was carried out to examine the effects of parameter changes on the optimal solution. The following parameter values were considered: $a = 40$, $b = 0.4$, $\beta = 0.5$, $\gamma_1 = 0.35$, $\gamma_2 = 0.4$, $\theta_0 = 0.3$, $\theta_1 = 0.1$, $\xi = 0.001$, $\lambda = 0.05$, $\theta(\psi) = \theta_1 + (\theta_0 - \theta_1)e^{-\xi\psi Q^{-\lambda}}$, $S_0 = 200$, $S_1 = 50$, $h = 0.1$, $c = 40$, $c_d = 0.2$, and $S_c = 200$. The parameters related to the BA on the basis of some preliminary parametric studies were as follows: $f_i = U(0, 3)$, $A_i = U(0.8, 1)$, $r_i = U(0.5, 1)$, $p = 0.7$, $\lambda = 5$, $\alpha = 0.98$, $\gamma = 0.02$, and $N = 100$. By applying the HBA, the following results were obtained in the static environment.

From Table 2, one can observe that investment in preservation technology is always profitable for the retailer. The retailer can earn more profit by providing rebates according to demand variations when applying market-skimming pricing. The retailer sets the highest price initially, and then provides considerable rebates on the basis of that initial price. In this situation, the retailer can earn as much profits as possible at the beginning of the replenishment cycle. This strategy gives the retailer the flexibility to invest more in preservation technology. High rebate amounts offered during the third stage enhance demand considerably and hence flow of product. Moreover, the length of the replenishment cycle is also greater under preservation technology investment; therefore, retailers benefit from making relatively long-term plans. Results for the dynamic environment are given in Table 3.

Table 3 Optimal price of the retailer, cycle time, amount of rebate, PTI, order quantity, and profit in dynamic environment

		p	t_1	t_2	T	R_1	R_2	ψ	Q	π
π^{d1}	With PTI	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13 \cosh 0.24t + 657.88 \sinh 0.24t$	4171.56	692.19	1417.38
	Without PTI	157.38	1.19	3.60	5.83	$106.55 - 1.84e^{0.3t}$	$128.21 - 226.18 \cosh 0.46t + 231.36 \sinh 0.46t$	–	344.71	471.13
π^{d2}	With PTI	114.16	4.57	9.07	11.19	$54.36 - 0.96e^{0.11t}$	–	3935.96	708.07	1232.70
	Without PTI	104.39	0.71	3.17	5.37	$42.18 - 9.47e^{0.3t}$	$42.18 - 9.47e^{0.3t}$	–	301.06	541.669

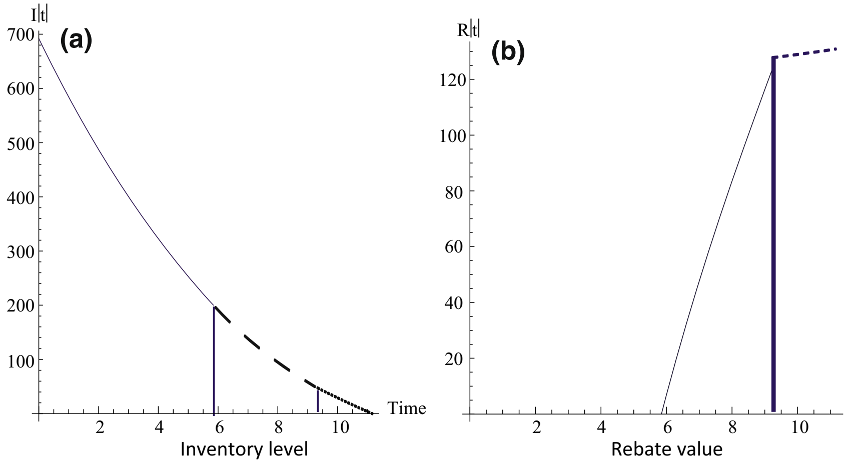

Fig. 1 **a** Inventory level. **b** Rebate value

As in the static environment, investment in preservation technology is also profitable for the retailer in the dynamic environment. Moreover, by comparing results in Table 2 with those in Table 3, one can see that the dynamic rebate is more profitable for the retailer. Graphical representation of the rebate rates and inventory levels are depicted in Fig. 1a, b.

Figure 1a, b show that the retailer can accelerate the rate of change of inventory levels by providing rebates. The rebate value increases steadily as time passes. At the beginning of the replenishment cycle, the on-hand inventory level is relatively high, and to acquire as much profit as possible, the retailer sets a high price. Then, to encroach on the market, the retailer increases the rebate amounts. In the second time interval, demand decreases with DSL; therefore, to keep the demand as high as possible, the retailer provides a high value for the rebate. In a situation similar to the static rebate scenario, the retailer can establish a longterm replenishment plan when investing in preservation technology. Therefore, preservation technology investment not only reduces the product loss, but the retailer can also compensate for frequent replenishment costs. The rate of rebate becomes steady in the third phase. It is also consistent with market-skimming pricing. The retailer tries to earn as much profit as possible when the demand is comparatively high. The results also show the impact of preservation technology investment on the optimal order quantity, which is reduced significantly in the absence of preservation technology investment. Tables 2 and 3 also show that the profit of the retailer is maximized when the rebate amount differs by time interval. After these trends were found, the sensitivity analysis with respect to system parameters was carried out to identify retailer preferences. When the value of one parameter varies, all others remained unchanged. The results are given in Tables 4 and 5.

On the basis of the computational results, we can obtain the following managerial insights:

1. When the values of parameters a, S_0, and b increased, the selling price p increased and the reserve trend was seen for changes in parameters β, θ_1, S_1, λ, and γ_2. It makes sense that the market demand and stock sensitivity have strong and positive effects upon the optimal selling price. As S_0 increases, the demand for the product increases, and therefore, the retailer can charge a higher price. Results also show that pricing decisions of the retailer were significantly influenced by the threshold value of the minimum deterioration rate. As

Table 4 Sensitivity analysis of π^{d1}

	P	t_1	t_2	T	R_1	R_2	π^{d1}	Q	ψ
$a = 50$	170.37	5.96	8.81	10.25	$108.33 + 9.47e^{0.11t}$	$130.62 - 706.62\text{Cos}h0.24t + 729.16\text{Sin}h0.24t$	2255.59	758.70	4312.83
$a = 45$	167.31	5.91	9.01	10.66	$111.76 + 7.04e^{0.11t}$	$133.43 - 676.16\text{Cos}h0.24t + 694.12\text{Sin}h0.24t$	1820.95	725.52	4244.67
$a = 40$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$a = 35$	159.96	5.75	9.48	11.80	$117.12 + 2.56e^{0.11t}$	$137.66 - 610.48\text{Cos}h0.24t + 620.41\text{Sin}h0.24t$	1047.81	658.83	4095.07
$a = 30$	155.56	5.63	9.76	12.68	$118.91 + 0.54e^{0.11t}$	$138.95 - 575.24\text{Cos}h0.24t + 581.75Sinh0.24t$	715.24	625.54	4017.42
$b = 0.50$	193.34	6.82	10.04	11.91	$147.01 + 5.13e^{0.11t}$	$172.35 - 1171.64Cosh0.26t + 1182.76\text{Sin}h0.26t$	2591.37	840.98	4438.01
$b = 0.45$	180.05	6.36	9.66	11.56	$130.81 + .4.94e^{0.11t}$	$154.02 - 876.44\text{Cos}h0.25t + 888.82\text{Sin}h0.25t$	1976.87	766.51	4317.25
$b = 0.4$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$b = 0.35$	148.09	5.27	8.77	10.75	$99.13 + 4.27e^{0.11t}$	$118.06 - 462.18\text{Cos}h0.23t + 477.02\text{Sin}h0.23t$	914.51	616.59	3990.4
$b = 0.30$	137.35	4.51	8.47	10.49	$89.65 + 0.56e^{0.11t}$	$105.95 - 310.45\text{Cos}h0.22t + 320.06\text{Sin}h0.27t$	494.59	517.98	3675.43
$\beta = 0.60$	145.65	3.41	9.71	12.23	$123.37 + 0.74e^{0.12t}$	$143.48 - 495.55\text{Cos}h0.25t + 501.60\text{Sin}h0.25t$	473.88	577.57	3239.13
$\beta = 0.55$	157.43	4.43	9.59	12.21	$118.14 + 1.39e^{0.11t}$	$138.40 - 586.38\text{Cos}h0.24t + 594.32\text{Sin}h0.24t$	852.00	637.19	3982.64
$\beta = 0.5$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$\beta = 0.45$	169.54	8.09	8.90	10.40	$109.50 + 8.69e^{0.11t}$	$131.60 - 700.73\text{Cos}h0.24t + 721.79\text{Sin}h0.24t$	2119.87	750.63	4346.87
$\beta = 0.40$	183.84	12.20	8.84	10.24	$108.72 + 9.06e^{0.11t}$	$125.30 - 742.70\text{Cos}h0.24t + 764.98\text{Sin}h0.24t$	2997.57	784.43	4429.57
$\gamma_1 = 0.400$	163.82	5.71	9.09	10.72	$110.75 + 4.79e^{0.11t}$	$135.75 - 624.20\text{Cos}h0.24t + 638.35\text{Sin}h0.24t$	1477.19	677.75	4122.31
$\gamma_1 = 0.375$	163.84	5.77	9.16	10.92	$112.60 + 4.77e^{0.11t}$	$135.77 - 633.55\text{Cos}h0.24t + 647.51\text{Sin}h0.24t$	1449.10	684.55	4146.05
$\gamma_1 = 0.35$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$\gamma_1 = 0.325$	163.88	5.93	9.31	11.44	$117.15 + 4.70e^{0.11t}$	$135.82 - 656.17\text{Cos}h0.24t + 669.69\text{Sin}h0.24t$	1381.29	700.82	4198.89
$\gamma_1 = 0.300$	163.90	6.02	9.41	11.77	$120.00 + 4.66e^{0.11t}$	$135.85 - 670.02\text{Cos}h0.24t + 683.29\text{Sin}h0.24t$	1339.84	710.65	4227.99
$\gamma_2 = 0.450$	154.31	5.66	8.65	10.47	$103.12 + 6.08e^{0.11t}$	$119.90 - 575.61\text{Cos}h0.24t + 593.54\text{Sin}h0.24t$	1539.90	706.23	4214.08
$\gamma_2 = 0.425$	158.70	5.75	8.91	10.76	$108.45 + 5.69e^{0.11t}$	$127.10 - 609.42\text{Cos}h0.24t + 625.62\text{Sin}h0.24t$	1489.60	701.29	4203.79

Table 4 continued

	P	t_1	t_2	T	R_1	R_2	π^{d1}	Q	ψ
$\gamma_2 = 0.4$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\mathrm{Cos}h0.24t + 657.88\mathrm{Sin}h0.24t$	1417.38	692.19	4171.56
$\gamma_2 = 0.375$	169.18	5.93	9.60	11.64	$121.18 + 3.59e^{0.11t}$	$145.50 - 680.61\mathrm{Cos}h0.24t + 691.71\mathrm{Sin}h0.24t$	1320.80	680.84	4120.44
$\gamma_2 = 0.350$	174.57	6.00	10.02	12.21	$127.71 + 2.26e0.11t$	$156.29 - 719.17\mathrm{Cos}h0.24t + 727.47\mathrm{Sin}h0.24t$	1198.90	667.51	4048.73
$S_0 = 225$	181.20	6.33	10.27	12.37	$135.77 + 3.57e^{0.11t}$	$155.30 - 780.82\mathrm{Cos}h0.23t + 790.92\mathrm{Sin}h0.23t$	1777.92	805.81	4423.05
$S_0 = 250$	198.02	6.82	11.29	13.59	$156.20 + 2.61e^{0.11t}$	$174.24 - 940.94\mathrm{Cos}h0.23t + 948.33\mathrm{Sin}h0.23t$	2147.33	931.94	4651.10
$S_0 = 200$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\mathrm{Cos}h0.24t + 657.88\mathrm{Sin}h0.24t$	1417.38	692.19	4171.56
$S_0 = 175$	147.05	5.33	8.22	10.05	$94.29 + 5.23e^{0.11t}$	$116.87 - 523.00\mathrm{Cos}h0.24t + 540.45\mathrm{Sin}h0.24t$	1062.74	584.40	3865.11
$S_0 = 150$	135.68	4.70	7.27	9.34	$80.47 + 1.11e^{0.12t}$	$104.07 - 399.05\mathrm{Cos}h0.25t + 414.50\mathrm{Sin}h0.25t$	748.25	460.38	3396.28
$S_1 = 70$	157.29	5.65	8.22	10.35	$95.31 + 6.68e^{0.11t}$	$128.41 - 630.35\mathrm{Cos}h0.24t + 651.24\mathrm{Sin}h0.24t$	1534.35	703.90	4213.84
$S_1 = 60$	158.62	5.77	8.64	10.57	$102.64 + 7.26e^{0.11t}$	$129.90 - 644.60\mathrm{Cos}h0.24t + 663.72\mathrm{Sin}h0.24t$	1480.64	696.08	4177.90
$S_1 = 50$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\mathrm{Cos}h0.24t + 657.88\mathrm{Sin}h0.24t$	1417.38	692.19	4171.56
$S_1 = 40$	168.35	5.91	9.94	11.85	$125.88 + 2.51e^{0.11t}$	$140.85 - 644.07\mathrm{Cos}h0.24t + 653.35\mathrm{Sin}h0.24t$	1338.28	682.17	4120.33
$S_1 = 30$	171.26	5.98	10.29	12.14	$135.10 + 0.65e^{0.12t}$	$144.10 - 605.85\mathrm{Cos}h0.24t + 611.68\mathrm{Sin}h0.24t$	1240.19	651.56	3484.99
$\theta_0 = 0.40$	163.90	5.96	9.35	11.28	$114.77 + 4.69e^{0.11t}$	$135.85 - 660.68\mathrm{Cos}h0.24t + 674.13\mathrm{Sin}h0.24t$	1367.25	703.91	4791.00
$\theta_0 = 0.35$	163.89	5.91	9.30	11.23	$114.75 + 4.71e^{0.11t}$	$135.81 - 653.25\mathrm{Cos}h0.24t + 666.84\mathrm{Sin}h0.24t$	1389.73	698.66	4512.62
$\theta_0 = 0.4$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\mathrm{Cos}h0.24t + 657.88\mathrm{Sin}h0.24t$	1417.38	692.19	4171.56
$\theta_0 = 0.25$	163.83	5.76	9.15	11.07	$114.67 + 4.77e^{0.11t}$	$135.76 - 632.34\mathrm{Cos}h0.24t + 646.31\mathrm{Sin}h0.24t$	1453.25	683.76	3731.26
$\theta_0 = 0.20$	163.78	5.64	9.02	10.95	$114.61 + 4.82e^{0.11t}$	$135.70 - 615.70\mathrm{Cos}h0.24t + 629.99\mathrm{Sin}h0.24t$	1504.27	671.72	3109.44
$\theta_1 = 0.150$	160.62	3.41	6.40	8.35	$110.62 + 2.77e^{0.17t}$	$132 - 420.94\mathrm{Cos}h0.31t + 432.64\mathrm{Sin}h0.31t$	910.085	537.61	2920.8
$\theta_1 = 0.125$	160.8	4.43	7.54	9.38	$110.90 + .4.77e^{0.14t}$	$132.26 - 514.34\mathrm{Cos}h0.27t + 528.46\mathrm{Sin}h0.27t$	1140.61	610.56	3553.86

Table 4 continued

	P	t_1	t_2	T	R_1	R_2	π^{d1}	Q	ψ
$\theta_1 = 0.1$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$\theta_1 = 0.075$	167.62	8.09	11.83	13.85	$119.44 + 4.72e^{0.14t}$	$140.18 - 859.28\text{Cos}h0.20t + 872.61\text{Sin}h0.20t$	1741.42	803.48	4855.88
$\theta_1 = 0.050$	172.44	12.20	16.39	18.53	$125.60 + 4.72e^{0.05t}$	$145.91 - 1285.55\text{Cos}h0.16t + 1298.40\text{Sin}h0.16t$	2124.68	973.41	5679.97
$\lambda = 0.07$	163.70	5.80	9.17	11.09	$114.52 + 4.72e^{0.11t}$	$135.61 - 641.75\text{Cos}h0.24t + 655.40\text{Sin}h0.24t$	1366.13	691.63	4552.5
$\lambda = 0.06$	163.78	5.82	9.21	11.13	$114.62 + .4.73e^{0.11t}$	$135.71 - 643.08\text{Cos}h0.24t + 656.79\text{Sin}h0.24t$	1392.35	691.99	4359.22
$\lambda = 0.05$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$\lambda = 0.04$	163.93	5.86	9.26	11.19	$114.81 + 4.74e^{0.11t}$	$135.88 - 644.90\text{Cos}h0.24t + 658.70\text{Sin}h0.24t$	1441.26	692.23	3989.71
$\lambda = 0.03$	164.00	5.88	9.29	11.22	$114.89 + 4.75e^{0.11t}$	$135.96 - 645.46\text{Cos}h0.24t + 659.31\text{Sin}h0.24t$	1464.03	692.42	3813.8

Table 5 Sensitivity analysis π^{s1}

	p	t_1	t_2	T	R_1	R_2	π^{s1}	Q	ψ
$a = 50$	150.09	5.70	8.93	10.58	93.56	44.86	2282.92	806.65	4356.61
$a = 45$	145.12	5.67	9.09	10.92	92.74	45.64	1811.86	782.72	4322.28
$a = 40$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$a = 35$	135.10	5.55	9.49	11.81	90.09	47.02	1045.07	730.04	4236.83
$a = 30$	130.05	5.46	9.67	12.46	88.03	48.09	712.63	700.51	4182.87
$b = 0.50$	161.73	6.59	9.64	11.04	121.73	68.77	2704.82	970.26	4745.70
$b = 0.45$	151.40	6.16	9.49	11.25	111.40	58.06	2010.85	863.40	4529.31
$b = 0.4$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$b = 0.35$	128.38	4.96	9.00	11.41	70.80	34.69	912.52	650.57	4001.33
$b = 0.30$	114.89	4.14	8.57	11.52	49.23	22.69	510.62	542.72	3673.75
$\beta = 0.60$	112.71	4.73	9.10	11.99	63.65	28.12	588.28	665.40	3954.44
$\beta = 0.55$	125.60	5.22	9.23	11.64	77.77	37.25	943.06	714.43	4133.30
$\beta = 0.5$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$\beta = 0.45$	157.21	5.96	9.29	11.02	105.92	56.31	2033.85	794.82	4408.58
$\beta = 0.40$	178.04	6.24	9.24	10.72	121.35	67.47	2867.55	826.22	4510.52
$\gamma_1 = 0.400$	141.37	5.60	9.07	11.07	86.97	48.49	1459.91	745.59	4249.22
$\gamma_1 = 0.375$	140.79	5.61	9.18	11.19	89.23	47.43	1436.94	751.27	4265.82
$\gamma_1 = 0.35$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$\gamma_1 = 0.325$	139.37	5.62	9.39	11.44	94.11	45.21	1385.20	763.66	4299.92
$\gamma_1 = 0.300$	138.48	5.62	9.49	11.56	96.70	43.79	1355.86	770.38	4317.02
$\gamma_2 = 0.450$	142.71	5.60	9.07	11.07	96.36	48.04	1466.72	744.17	4262.02
$\gamma_2 = 0.425$	141.50	5.61	9.18	11.19	94.10	47.37	1439.73	750.87	4273.65
$\gamma_2 = 0.4$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$\gamma_2 = 0.375$	138.57	5.62	9.39	11.44	88.85	45.15	1383.97	763.23	4288.48
$\gamma_2 = 0.350$	136.78	5.62	9.49	11.56	85.73	43.36	1355.45	768.36	4289.42
$S_0 = 250$	159.85	6.64	10.76	12.79	119.85	72.48	2231.50	1083.23	4986.59
$S_0 = 225$	150.46	6.16	10.07	12.00	110.46	59.92	1807.99	913.57	4652.38
$S_0 = 200$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$S_0 = 175$	129.33	5.01	8.35	10.60	69.15	32.87	1072.01	611.68	3862.86
$S_0 = 150$	118.27	4.30	7.23	9.73	46.76	19.68	781.75	475.97	3370.87
$S_1 = 70$	138.53	5.26	8.21	10.78	70.80	31.29	1567.69	718.47	4150.52
$S_1 = 60$	139.37	5.45	8.74	11.06	81.19	38.67	1487.50	738.62	4338.15
$S_1 = 0.4$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$S_1 = 40$	140.69	5.77	9.83	11.57	100.69	54.49	1340.28	774.85	4338.15
$S_1 = 30$	140.43	5.91	10.39	11.91	105.43	62.03	1265.17	792.91	4378.53
$\theta_0 = 0.40$	140.48	5.73	9.42	11.48	91.09	46.31	1362.56	768.91	4898.97
$\theta_0 = 0.35$	140.32	5.68	9.36	11.41	91.33	46.37	1384.76	763.72	4622.02
$\theta = 0.3$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$\theta_0 = 0.25$	139.87	5.55	9.19	11.20	91.98	46.52	1447.71	748.89	3844.86
$\theta_0 = 0.20$	139.50	5.43	9.05	11.03	92.46	46.63	1498.49	736.83	3226.58
$\theta_1 = 0.150$	134.21	3.18	6.52	8.76	72.59	34.50	913.64	563.64	2915.77

Table 5 continued

	p	t_1	t_2	T	R_1	R_2	π^{s1}	Q	ψ
$\theta_1 = 0.125$	137.40	4.19	7.71	9.86	81.83	40.30	1131.82	648.27	3579.04
$\theta_1 = 0.4$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$\theta_1 = 0.075$	142.42	7.81	11.57	13.46	102.24	53.23	1776.63	908.98	5077.00
$\theta_1 = 0.050$	143.65	11.63	15.40	17.35	103.65	60.26	2263.37	1149.83	6051.63
$\lambda = 0.07$	140.24	5.58	9.26	11.32	90.69	46.01	1359.67	755.18	4675.96
$\lambda = 0.06$	140.18	5.60	9.27	11.32	91.17	46.23	1386.47	756.36	4476.64
$\lambda = 0.05$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$\lambda = 0.04$	140.07	5.64	9.30	11.31	92.04	46.64	1436.66	757.99	4094.63
$\lambda = 0.03$	140.01	5.65	9.31	11.31	92.43	46.83	1460.12	758.48	3912.43

the threshold value decreases, the retailer needs to invest more in preservation technology, hence the price of the product increases.

2. When the values of parameters a, γ_1, γ_2, S_1, and θ_1 increased, the replenishment time decreased. However, the reverse trend was found for parameters β, b, S_0, and θ_0. The replenishment cycle time was greatly influenced by θ_1. The deterioration rate decreased as θ_1 decreased. Therefore, the replenishment cycle time increased. If the price sensitivity increases, then the retailer cannot charge a higher price to compensate for fixed costs, so the retailer needs to extend the replenishment cycle time.
3. The preservation technology investment decision of the retailer was positively correlated with a, b, γ_2, S_0, S_1, θ_0, and λ and negatively related with β, γ_1, and θ_1. In contrast to findings in the literature, we found that the retailer cannot reduce the deterioration rate completely. We also found that the threshold value for the minimum deterioration rate significantly influenced the retailer's decision on the optimal investment in preservation technology. A high threshold value of θ_1 discouraged the retailer from investing in preservation technology in every case. This result is quite realistic. If the retailer cannot reduce the deterioration rate significantly, then the investment on deterioration is not always profitable for the retailer. We also found that the retailer should invest more in preservation technology and order more to reduce losses caused by deterioration and to satisfy market demand, when the deterioration rate is relatively high.
4. The rebate rates R_1 and R_2 were positively correlated with b, γ_2, and S_0, and negatively correlated with β, θ_0, and θ_1. A mixed trend was found for the parameters a and γ_1. After time t_2, the inventory level reached S_1 and as a consequence the demand becomes steady. The retailer needs to provide huge rebate amounts to accelerate the flow of the product and in this way reduce the effect of deterioration. The rebate sensitivity for the final time period was crucial for determining the appropriate rebate for the previous period.
5. Finally, the profit of the retailer was positively correlated with a, b, S_0, θ_0, and λ, and it was negatively correlated with β, γ_1, γ_2, S_1, and θ_1. Overall, a large b implies more demand, which shortens the sales period and increases the total profit per unit time. As a consequence, taking measures to enhance b, such as adopting more effective displays, improving the shopping environment, and so forth, benefits the retailer. The retailer can enhance profit by reducing the rate of deterioration and thus provide preservation technology investment. The retailer choices for rebates in static and dynamic environments are influenced by parameters b, β, γ_1, and θ_1. The retailer profits in both dynamic and static environments are depicted in Fig. 2.

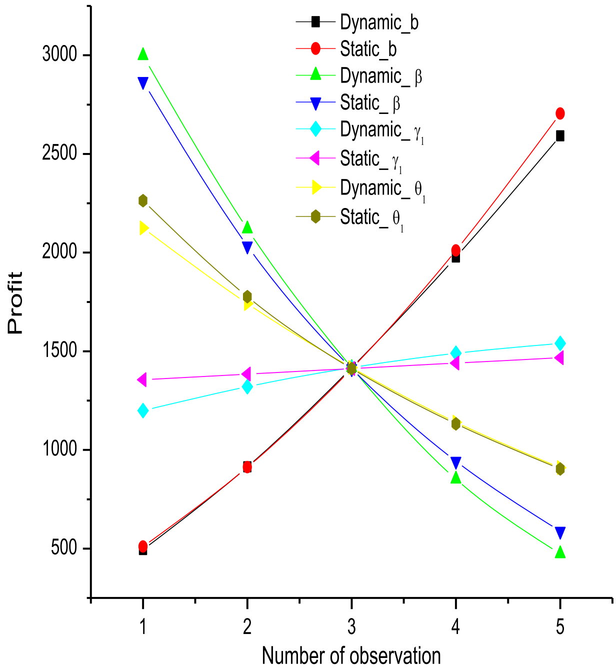

Fig. 2 π^{d1} versus π^{s1}

From Fig. 2, one can observe that display stock, price, and rebate sensitivities during the final time period and the threshold value of the deterioration rate are key parameters for making decision about rebate strategy. If the display stock sensitivity is high or price sensitivity is low, then the retailer should adopt a static rebate to obtain maximum profit, otherwise the retailer should choose a dynamic rebate.

From Table 6, one can see that the replenishment cycle increased while S_c increased, and the reverse trend was followed for the other parameters. That retailer profit decreases with increased values for cost parameters follows an intuitively derived conclusion. The numerical results also justify the intuitive conclusion. The higher the value of c, the higher the sales price, which implies that, when the purchasing cost c increases, the retailer will augment the increased purchasing cost with the sales price. In addition, when the purchasing cost c increases, the replenishment cycle increases and the total profit per unit time decreases. For a large c, the retailer will extend the replenishment cycle and increase the sales price to compensate for the increased purchasing cost. Moreover, the purchase cost plays a decisive role for rebate strategy selection.

7 Summary and concluding remarks

In this study, we considered a decision-making problem for a retailer determining sale price, rebate, investment in preservation technology, and replenishment cycle time, simultaneously under price, displayed stock level, and rebate-induced demand by maximizing the total profit per unit time. Models were formulated under static and dynamic environments. To solve the

Table 6 Sensitivity analysis with respect to cost parameters for π^{d1} and π^{s1}

	P	t_1	t_2	T	R_1	R_2	π^{d1}	Q	ψ
$c = 50$	168.67	4.96	8.66	10.98	$120.55 + 0.95e^{0.11t}$	$141.21 - 511.34\text{Cos}h0.24t + 522.62\text{Sin}h0.24t$	812.85	580.85	4100.93
$c = 45$	166.34	5.39	8.92	11.03	$117.72 + 2.90e^{0.11t}$	$138.58 - 571.33\text{Cos}h0.24t + 584.02\text{Sin}h0.24t$	1101.69	632.38	4136.22
$c = 40$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$c = 35$	161.25	6.35	9.60	11.38	$111.54 + 6.41e^{0.11t}$	$132.86 - 735.02\text{Cos}h0.24t + 749.44\text{Sin}h0.24t$	1761.22	766.17	4207.02
$c = 30$	158.51	6.92	10.04	11.69	$108.22 + 7.88e^{0.11t}$	$129.78 - 852.75\text{Cos}h0.24t + 867.37\text{Sin}h0.24t$	2135.34	849.74	4242.58
$h = 0.150$	163.79	5.80	9.19	11.11	$114.86 + 4.68e^{0.11t}$	$135.95 - 637.43\text{Cos}h0.24t + 651.30\text{Sin}h0.24t$	1405.05	687.28	4156.55
$h = 0.125$	163.83	5.82	9.21	11.14	$114.79 + 4.71e^{0.11t}$	$135.87 - 640.76\text{Cos}h0.24t + 654.56\text{Sin}h0.24t$	1411.20	689.72	4164.03
$h = 0.1$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$h = 0.075$	163.90	5.87	9.26	11.19	$114.64 + 4.76e^{0.11t}$	$135.72 - 647.53\text{Cos}h0.24t + 661.23\text{Sin}h0.24t$	1423.58	694.68	4179.12
$h = 0.050$	163.93	5.89	9.29	11.21	$114.57 + 4.79e^{0.11t}$	$135.65 - 650.99\text{Cos}h0.24t + 664.62\text{Sin}h0.24t$	1429.79	697.19	4186.74
$c_d = 3.0$	163.74	5.76	9.15	11.07	$115.06 + 4.60e^{0.11t}$	$136.16 - 631.57\text{Cos}h0.24t + 645.55\text{Sin}h0.24t$	1390.36	682.74	4171.96
$c_d = 2.5$	163.80	5.80	9.19	11.12	$114.89 + 4.67e^{0.11t}$	$135.98 - 637.78\text{Cos}h0.24t + 651.64\text{Sin}h0.24t$	1403.83	687.42	4171.77
$c_d = 2$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$c_d = 1.5$	163.92	5.89	9.28	11.21	$114.54 + 4.80e^{0.11t}$	$135.62 - 650.64\text{Cos}h0.24t + 664.27\text{Sin}h0.24t$	1431.02	697.05	4171.33
$c_d = 1.0$	163.99	5.93	9.33	11.25	$114.36 + 4.86e^{0.11t}$	$135.44 - 657.32\text{Cos}h0.24t + 670.84\text{Sin}h0.24t$	1444.74	702.00	4171.09
$S_c = 250$	163.86	5.86	9.26	11.17	$114.72 + 4.73e^{0.11t}$	$135.80 - 645.73\text{Cos}h0.24t + 659.45\text{Sin}h0.24t$	1412.90	693.33	4176.95
$S_c = 225$	163.86	5.85	9.24	11.17	$114.72 + 4.73e^{0.11t}$	$135.80 - 644.93\text{Cos}h0.24t + 658.66\text{Sin}h0.24t$	1415.14	692.76	4174.26
$S_c = 200$	163.86	5.84	9.24	11.16	$114.72 + 4.73e^{0.11t}$	$135.79 - 644.13\text{Cos}h0.24t + 657.88\text{Sin}h0.24t$	1417.38	692.19	4171.56
$S_c = 175$	163.86	5.84	9.23	11.16	$114.71 + 4.74e^{0.11t}$	$135.80 - 643.32\text{Cos}h0.24t + 657.09\text{Sin}h0.24t$	1419.62	691.61	4168.85
$S_c = 150$	163.86	5.83	9.22	11.15	$114.71 + 4.74e^{0.11t}$	$135.79 - 642.52\text{Cos}h0.24t + 656.30\text{Sin}h0.24t$	1421.86	691.04	4166.15

Table 6 continued

	P	t_1	t_2	T	R_1	R_2	π^{s1}	Q	ψ
$c = 50$	143.94	4.73	8.78	11.32	83.05	43.89	798.28	633.45	4186.52
$c = 45$	142.09	5.16	9.01	11.26	87.50	45.17	1091.53	691.04	4235.19
$c = 40$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$c = 35$	138.06	6.13	9.63	11.48	95.48	47.72	1761.21	835.37	4329.37
$c = 30$	135.89	6.71	10.04	11.74	99.17	49.03	2140.87	930.09	4375.26
$h = 0.150$	140.07	5.57	9.24	11.28	91.29	46.28	1399.11	751.34	4265.01
$h = 0.125$	140.10	5.60	9.26	11.30	91.45	46.35	1405.60	754.30	4273.87
$c = 0.1$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$h = 0.075$	140.16	5.65	9.31	11.34	91.78	46.53	1418.67	760.31	4291.71
$h = 0.050$	140.19	5.67	9.33	11.36	91.94	46.61	1425.25	763.36	4300.70
$c_d = 3.0$	140.03	5.53	9.21	11.25	90.95	46.01	1388.81	745.83	4277.64
$c_d = 2.5$	140.07	5.58	9.25	11.28	91.28	46.27	1397.91	751.51	4280.19
$c_d = 2$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$c_d = 1.5$	140.17	5.67	9.32	11.35	91.95	46.61	1426.46	763.19	4285.36
$c_d = 1.0$	140.22	5.71	9.37	11.38	92.28	46.77	1440.91	769.19	4287.96
$S_c = 250$	140.16	5.63	9.30	11.33	91.57	46.43	1407.70	758.43	4287.63
$S_c = 225$	140.14	5.63	9.29	11.32	91.59	46.43	1409.91	757.86	4285.21
$S_c = 200$	140.13	5.62	9.29	11.32	91.62	46.44	1412.12	757.29	4282.77
$S_c = 175$	140.11	5.62	9.28	11.31	91.64	46.44	1414.33	756.72	4280.33
$S_c = 150$	140.09	5.61	9.27	11.30	91.66	46.45	1416.54	756.15	4277.89

models in the dynamic environment, Pontryagin's maximum principle was employed to find the optimal rebate. An HBA was developed to find the solutions to the models.

The present study has contributed to the literature in the following ways: First, the effect of a dynamic rebate on the initial price of a product had not been analyzed, and we found that such a pricing scheme is sometimes more profitable than the static rebate. The present study has contributed to the literature in the following ways: First, the effect of a dynamic rebate on the initial price of a product had not been analyzed, and we found that such a pricing scheme is sometimes more profitable than use of a static rebate. By determining price-and-rebate pairs, the retailer can skim as much profits as possible at the beginning of the replenishment cycle and then adjust the rebate amount as time elapses. The pair creates a price-anchoring effect in the consumers' minds that comes into play when they compare subsequent offerings with the high initial price. This is perhaps the first study in which price, dynamic or static rebate amount, preservation technology investment, and replenishment time were determined simultaneously to establish a pragmatic framework for the retailers selling deteriorating items. Second, the analytical solutions related to the dynamic and static rebates can serve as powerful tools for making replenishment decisions by a monopolistic retailer. Computational experiments showed that the price differentiation in three different phases for a generalized stock-dependent demand proved profitable for the retailer. The results also demonstrate that price sensitivity in the last phase, when demand becomes independent from display-stock, plays an important role in the optimal replenishment decision. Third, retailers need to decide preservation technology investment according to the nature of the products. The optimal order quantity and preservation technology investment decision are associated with each other. The results indicated that the retailer needs to provide a high rebate amount, especially in the middle phase, for products with high deterioration rates. Finally, creating a hybrid algorithm to solve complex optimization problems is a growing avenue of research. The developed algorithm, HBA, performed efficiently to solve the standard non-linear functions and the developed models. In time, optimization tools can be useful for finding solutions to non-linear constraint optimization problems.

Despite its importance and contribution, the present study has some limitations. For instance, the rebate redemption effect is ignored; therefore, an extended model to study the effect of the redemption rate and a survey to estimate the range of parameter values would be worthwhile. The concepts addressed in the paper could also be advanced in several ways. One could extend the proposed model by incorporating trade credit financing as seen in (Das et al. 2015) or sales team effort (Cárdenas-Barrón and Sana 2014). One might expand a single-player optimal solution to multiple players integrated in a supply chain situation, as presented in (Saha and Goyal 2015) and study the effect of redemption. We formulated the proposed model under a single-replenishment setting, one may study the effect of rebate in multi-period setting as discussed in Dye and Yang (2016), Hsieh and Dye (2017).

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Appendix A

Ten popular benchmark functions are used to verify the performance of the new HBA and compared with that of standard BA. The solution obtained after each trial was recorded for computing mean values. The best solution, the worst solution, and the mean values are shown

Table 7 Performance of HBA and BA on some standard test functions

Function name	Algorithm	Min	Max	Mean
Eggholder function	HBA	– 959.641	– 934.851	−954.004
	BA	– 959.641	– 632.756	– 830.937
Holder Table function	HBA	– 19.209	– 19.206	– 19.208
	BA	– 19.209	– 16.118	– 17.549
Dropwave function	HBA	– 1.000	– 0.996	– 1.000
	BA	– 1.000	– 0.771	– 0.945
Schafer function N.4	HBA	0.500	0.500	0.500
	BA	0.500	0.500	0.500
Cross-in-Tray function	HBA	– 2.063	– 2.063	– 2.063
	BA	– 2.063	– 2.063	– 2.063
Michalewicz's function	HBA	– 1.801	– 1.801	– 1.801
	BA	– 1.797	– 1.686	– 1.722
Shubert function	HBA	– 186.731	– 186.728	– 186.731
	BA	– 186.731	– 78.509	– 174.358
Easom's function	HBA	– 1.000	– 0.925	– 0.995
	BA	– 1.000	0.000	– 0.157
Rosenbrock's function	HBA	0.000	0.007	0.000
	BA	0.000	2.148	0.298
Styblinski-Tang function	HBA	– 78.332	– 78.331	– 78.332
	BA	– 78.332	– 62.496	– 76.619

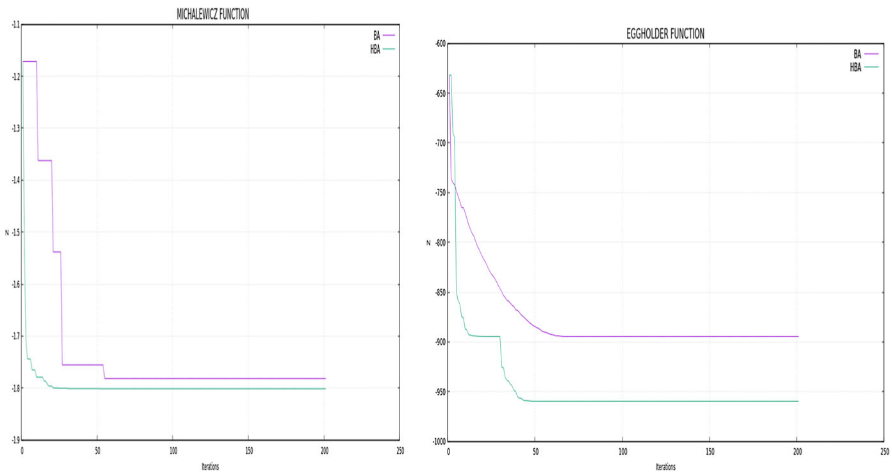

Fig. 3 Convergence evolutions of objective functions

in Table 7. Each algorithm was run 30 times with the maximum number of iterations set as 1000.

Additionally, Fig. 3 represents the evolution at each iteration for two classical benchmark functions.

From the above Table 7 and Fig. 3, one can find that the hybridization of BA with Mantegna Algorithm and Chaotic inertia function induced accelerated convergence rate and enabled HBA to traverse a large area of objective landscape, which in turn reduced the probability of premature convergence. The Chaotic inertia function made the trajectory of virtual bats more diverse and stabilized. The proportion of global search can also be set by changing the value of switch probability depending on the optimization problem of interest. Therefore, the hybridization and modifications makes the algorithm robust.

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