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Controllable production rates in a family production context

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We consider a single facility dedicated to the production of a family of items where the policy used is to cycle production through the items every T units of time. This is known as the Common Cycle or Rotation Schedule and is a special case of the Economic Lot Scheduling Problem (ELSP). We consider the case in which the production rates can be reduced from known maximum rates. Moreover, we assume that production rates can be changed during the production runs. We optimally partition the items into those with high and low holding costs. The former are initially produced to meet demand while the latter are always produced at their maximum rates. Numerical examples indicate savings almost twice as large as those reported in the literature.

1. Introduction

We consider a single facility dedicated to the production of a family of items. The objective is to find a schedule to minimize the setup costs and holding costs while satisfying demands for the items. This problem is well known as the Economic Lot Scheduling Problem (ELSP). A comprehensive review up to the 1970s is given by Elmaghraby (1978). Dobson (1987) overcame the feasibility issue by allowing time-varying production runs and gave an efficient heuristic.

In this paper, we restrict attention to the policy that cycles production through the items every T units of time. This policy is known as the Common Cycle Schedule or Rotation Schedule in the ELSP literature. See Jones and Inman (1989) for review of this approach and for conditions under which a rotation schedule is optimal. A Common Cycle Schedule is appropriate when the individual item setup cost is not too large relative to its annual dollar usage (see Silver and Peterson 1985).

Recently, researchers have considered manufacturing strategies that reduce production rates deliberately. This is consistent with the Just-In-Time manufacturing philosophy which has been successfully applied in many Japanese manufacturing companies. Silver (1990) developed procedures to find optimal production rates in a family production context where a Common Cycle policy is used. He showed that at most one item reduces its production rate and obtained a closed form solution for the optimal cycle length and the optimal production rates. Inman and Jones (1989) developed a version of the ELSP with change-over or reducible production rates. Gallego (1989) developed an efficient procedure to find optimal production rates for a lower bound of the ELSP. He showed that at most one item, typically the slowest, should reduce its production rate. Then he applied a time-varying heuristic with the optimal production rates to find a near-optimal feasible cyclic schedule. In all of the above papers, the production rate is set at the beginning of each production run and is not allowed to change during the actual run.

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The ELSP is often used to model discrete part manufacturing with small per unit processing times in that it treats production as if it were continuous. In such systems, it is often possible to insert arbitrary idle times between each consecutive unit of an item without incurring a new setup, this is what Inman and Jones (1989) call a 'change-over'. In this case, any production rate lower than the maximum can be achieved. In particular, any item can be produced at its demand rate (see Sheldon 1987). Inman and Jones (1989) suggested slowing the production rates uniformly over a run by inserting small and identical idle times between the production of each consecutive unit. Proposition 1 in Section 2 shows that this is not the right thing to do. In fact what is optimal is to first produce to meet demand and then produce at the maximum rate, i.e. have larger idle times initially and then none at all. Arizono *et al.* (1989) discussed the effects of varying production rates on inventory control and showed that the system where production rates are controllable is much more effective than an uncontrollable system.

The problem we consider here is a generalization of Silver (1990). That is, we consider the Common Cycle problem and allow production rates to be controlled during the production runs.

In Section 2, we introduce notation and state the assumptions. In Section 3, we formulate the problem and develop a method to obtain an optimal schedule. In Section 4, we compare our results with Silver's (1990) and Inman and Jones' (1989) using numerical examples. In Section 5, we extend our results to allow backorders.

2. Preliminaries

Data

the index for the products		$i=1,\ldots,m$;
setup times	s_i	$i=1,\ldots,m$;
setup costs	A_i	$i=1,\ldots,m$;
maximum production rates	p'_i	$i=1,\ldots,m$;
unit inventory holding costs	h'_i	$i=1,\ldots,m$;
constant demand rates	d'_i	$i=1,\ldots,m$.

Without loss of generality, we redefine the product units so all the demand rates are one. This is accomplished by setting $d_i \equiv d'_i/d'_i = 1$, $p_i \equiv p'_i/d'_i$ and $h_i \equiv h'_i d'_i$. The transformation maps many equivalent ELSPs to one that is easier to manipulate. For convenience relabel items, if necessary, in decreasing order of $\mathbf{h}$, i.e. $h_1 \geqslant h_2 \geqslant \ldots \geqslant h_m$. Let $\mathbf{r} \equiv (1/p_1, \ldots, 1/p_m)^T$, note that $r_i = 1/p_i = d'_i/p'_i$. Also, let $\mathbf{e}$ be the vector of ones in m space. Define $\kappa \equiv 1 - \mathbf{e}^T\mathbf{r}$. Note that κ is the long run proportion of time available for setups. For infinite horizon problems $\kappa > 0$ is a necessary condition for the existence of a feasible schedule. Dobson (1987) showed that if $\kappa > 0$, then any production sequence can be converted into a feasible cyclic schedule with time-varying production runs.

Assumptions

(*a*) No backorders are allowed.
(*b*) Demand rates are known and constant.
(*c*) Setup times are constant and sequence independent.
(*d*) A cycle is made through the entire family every T years.

(*e*) Production rates can be controllable at the beginning of the production runs and during the production runs.
(*f*) $h_1 \leqslant \sum_j h_j(1-r_j)/2\kappa$.

Assumption (*a*) is relaxed in Section 5. Assumption (*f*) is a technicality used in the proof of the existence of a global minimum. In practice, this assumption is reasonable and holds in most cases.

First, we prove the rather obvious:

Proposition 1. The optimal schedule that allows the production rates to be changed during the production runs (Schedule 1) has costs at most as high as the optimal schedule that allows the reduction only at the beginning of the production run (Schedule 2).

Proof. Suppose that in schedule 2 the production rate of item i is reduced from p_i to $\hat{p}_i$ and has a production run of $\hat{t}_i$ units of time. Let $t_i = \hat{t}_i(\hat{p}_i - 1)/(p_i - 1)$, see Fig. 1, and construct a new schedule by producing to meet demand for $x_i = \hat{t}_i - t_i$ units of time and then at the maximum rate p_i for t_i units of time. Then the new schedule has lower or equal holding costs and hence total cost. Note that the cost of this new schedule is at most as large as that of schedule 2.

A moment should convince the reader that optimal rotation schedules are of the following form: first, when the inventory of item i hits zero, reduce its production rate to meet demands for x_i units of time; then produce item i at its maximum rate for t_i units of time. The reason is that the most effective way, in terms of minimizing the average inventory level, to build up $(p-1)t$ units of inventory in $t+x$ units of time is by producing to meet demand for x units of time.

The problem can now be stated as follows. There is a single facility on which m distinct products are to be produced. We try to find a common cycle length T, production times for meeting demand $x_1, \ldots, x_m$, production times at maximum rates $t_1, \ldots, t_m$ and idle times $u_1, \ldots, u_m$ so that the cycle can be repeated indefinitely, demand is met and inventory and setup costs per unit time are minimized. In Proposition 2, we will show that there exist no idle times in the optimal schedule.

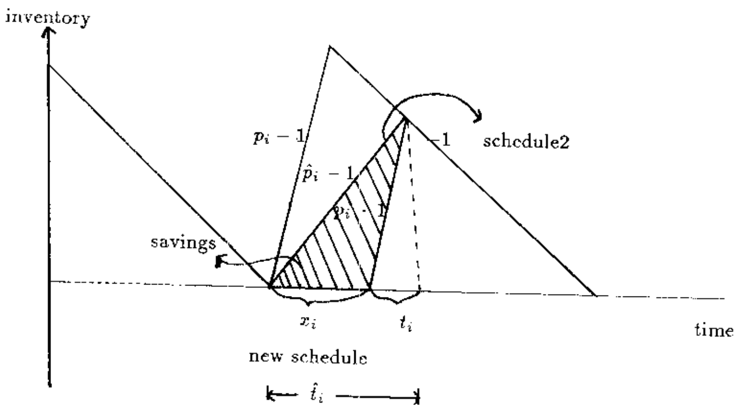

Figure 1. Comparison of schedule 1 and schedule 2.

3. Formulation and algorithm

The holding cost per cycle for item i is $\frac{1}{2}h_ip_i(p_i-1)t_i^2$ (see Fig. 2). Since $t_i=r_i(T-x_i)$, we can express the holding cost per cycle for item i as $\frac{1}{2}h_i\bar{r}_i(T-x_i)^2$ where $\bar{r}_i\equiv 1-r_i=1-1/p_i$. The objective is to minimize the average holding and setup cost per unit time. The only constraint is that the common cycle length must be large enough to accommodate setup times and production run times. That is, $T\geqslant\Sigma_{i=1}^m(s_i+x_i+t_i)$ which is equivalent to $\Sigma_is_i/T+\Sigma_i\bar{r}_ix_i/T\leqslant\kappa$. So, we can formulate the problem as follows.

$$\min_{x,T}\frac{1}{T}\sum_{i=1}^{m}[A_i+\tfrac{1}{2}h_i\bar{r}_i(T-x_i)^2]$$

$$\text{s.t.}\sum_i\frac{s_i}{T}+\sum_i\frac{\bar{r}_ix_i}{T}\leqslant\kappa \qquad (1)$$

$$x_i\geqslant 0 \qquad i=1,\ldots,m$$

First, we prove that there are no idle times in the optimal solution, so we have equality in Equation (1).

Proposition 2. The optimal solution of **P** always satisfies (1) as equality.

Proof. Suppose Equation (1) is not tight at the optimal solution. Increasing any x_i decreases the objective function contradicting optimality.

Proposition 3. The objective function of **P** is strictly convex in (x, T).

Proof. Follows directly from verifying that the Hessian matrix is positive definite.

(*Remark*) It is interesting to observe that convexity can also be shown by using the fact that $F(x)=f^2(x)/g(x)$ is convex on the convex set C in R^n whenever $f(x)$ is a nonnegative convex function and $g(x)$ is a positive concave function on the convex set C in R^n. See Drezner *et al.* (1990).

Even though the objective function is strictly convex in (x, T), a finite solution minimizing it may not be attained. For instance if $x_i=T\cdots$ constant for all i, then the objective function decreases to zero as $T\rightarrow\infty$. We show the existence of a unique global minimum by establishing that the level sets of the objective function are compact over the set of feasible solutions.

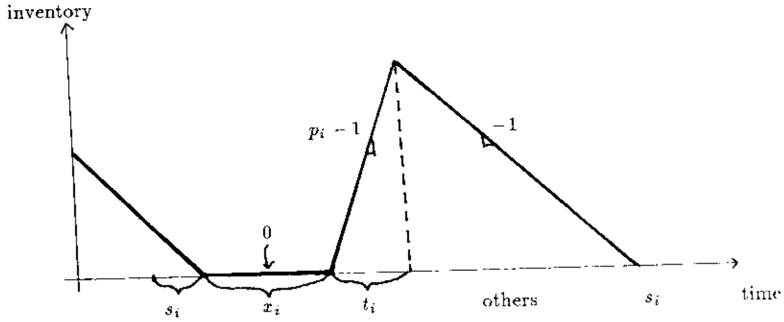

Figure 2. Inventory cost for a single product.

Theorem 1. Problem **P** has a unique global minimum.

Proof. See Appendix A.

Let λ and $\mu_i \geqslant 0$ be the Lagrange multipliers corresponding to Equation (1) and $x_i \geqslant 0$, respectively. Then the Karush–Kuhn–Tucker conditions for **P** are:

$$T\left(1+\frac{\mu_i}{h_i\bar{r}_i}-\frac{\lambda}{h_iT}\right)=x_i\geqslant 0 \text{ complementary slackness with } \mu_i\geqslant 0, \quad (2)$$

$$\sum_i A_i+\lambda\sum_i s_i+\lambda\sum_i \bar{r}_i x_i+\tfrac{1}{2}\sum_i h_i\bar{r}_i x_i^2-\tfrac{1}{2}\sum_i h_i\bar{r}_i T^2=0 \quad (3)$$

$$\sum_i s_i+\sum_i \bar{r}_i x_i-\kappa T=0. \quad (4)$$

Equation (2) can be replaced by

$$x_i=\left(T-\frac{\lambda}{h_i}\right)^+ \quad (5)$$

where $\{x\}^+=\max\{x,0\}$.
Equation (3) can be replaced by

$$f(T,\lambda)=\tfrac{1}{2}\sum_i h_i\bar{r}_i\left[\min\left(T,\frac{\lambda}{h_i}\right)\right]^2-\sum_i A_i-\lambda\sum_i s_i=0 \quad (6)$$

Equation (4) can be replaced by

$$g(T,\lambda)=\sum_i s_i+\sum_i \bar{r}_i\left(T-\frac{\lambda}{h_i}\right)^+-\kappa T=0. \quad (7)$$

We study the parametric solutions $T_1(\lambda)$ and $T_2(\lambda)$ of the functions $f(T,\lambda)$ and $g(T,\lambda)$ in Appendix B. We now present an algorithm to solve **P**.

Algorithm

Step 0 Start from an arbitrary $\lambda\geqslant\max\{\lambda^1, \lambda_m\}$.
(See Appendix B for definitions of λ^1 and λ_m.)

Step 1 (Compute $T_1(\lambda)$.)
Find a largest index j such that

$$f_j(\lambda)=\tfrac{1}{2}\sum_{i\leq j}\frac{\bar{r}_i}{h_i}\lambda^2+\tfrac{1}{2}\sum_{i>j}h_i\bar{r}_i\left(\frac{\lambda}{h_j}\right)^2-\sum_i A_i-\lambda\sum_i s_i<0,$$

and set

$$T_1(\lambda)=\left(\left[\sum_i A_i+\lambda\sum_i s_i-\tfrac{1}{2}\sum_{i\leq j}\frac{\bar{r}_i}{h_i}\lambda^2\right]\Big/\tfrac{1}{2}\sum_{i>j}h_i\bar{r}_i\right)^{1/2}$$

Step 2 (Compute $T_2(\lambda)$.)
Find a largest index k such that

$$g_k(\lambda)=\sum_{i\leq k}\bar{r}_i\left(\frac{\lambda}{h_k}-\frac{\lambda}{h_i}\right)+\sum_i s_i-\kappa\frac{\lambda}{h_k}<0,$$

and set

$$T_2(\lambda)=\left[\lambda\sum_{i\leq k}\frac{\bar{r}_j}{h_j}-\sum_i s_i\right]\Big/\left[\sum_{i\leq k}\bar{r}_j-\kappa\right].$$

Step 3 (Check stopping criteria.)
If $T_1(\lambda)=T_2(\lambda)$, stop. $\lambda^*=\lambda$. $T^*=T_1(\lambda^*)=T_2(\lambda^*)$.

$$\text{Optimal } x_i^*=T^*\left(1-\frac{\lambda^*}{h_iT^*}\right)^+ \text{ and } t_i^*=r_i\min\left\{T^*,\frac{\lambda^*}{h_i}\right\} \text{ for all } i.$$

If $T_1(\lambda)>T_2(\lambda)$, then increase λ. Go to *Step 1*.
If $T_1(\lambda)<T_2(\lambda)$, then decrease λ. Go to *Step 1*.

The items are partitioned into those with holding cost above the critical level λ^*/T^* and those below. The former are initially produced to meet demand, while the latter always produce at their maximum rates.

4. Numerical examples

First, we solve Silver's (1990) examples. The normalized data ordered in decreasing h_i are given in Table 1, the time unit is a year. We refer to the data in Table 1 as problem 1. Problem 2 is different only in p_4 which is changed to 8. Problem 3 is as Problem 1 except that each s_i is multiplied by 10.

In Table 2, we show the computational details for solving Problem 1 using the algorithm in the previous section. The optimal common cycle length for schedule 1 (ours) and schedule 2 (Silver's) need not be same. In Problem 1, the optimal common cycle length of schedule 1 is 0·200 while that of schedule 2 is 0·195. In Problem 1, the optimal solution of schedule 1 is x = (0·018, 0, 0, 0) and t = (0·091, 0·050, 0·030, 0·010).

Table 3 shows the computational results for these examples. The percentage shows the relative savings of each schedule compared to the usual Common Cycle approach. For Problem 3, there were no savings in either schedule. The savings of schedule 1 and schedule 2 come from the fact that we can reduce the production rates. The savings of schedule 1 is almost twice that of schedule 2. Additional savings of schedule 1 compared with schedule 2 come from the fact that we can control production rates during the production runs. The savings of schedule 1 and schedule 2 for Problem 1 and Problem 2 compared to no savings for Problem 3 indicate that production rates reductions may be very worthwhile for lowly utilized facilities. Interestingly, the opposite was observed in Gallego and Moon (1989, 1990) where setup time reduction was shown to be more profitable in highly utilized facilities.

We also solve Bombergers' (1966) famous 10 item problem with different multiples of the base demand rate. The normalized data with base demand rate ordered in decreasing h_i are given in Table 4, the time unit is a day.

Table 5 shows the computational results for this problem. We also compare our results to those obtained by Inman and Jones (1989). Even though our schedules are confined to simple Rotation Schedules, we obtained a significantly lower cost for the problem with the base demand rate.

Product number	s_i (year)	A_i (\$)	p_i (units/year)	h_i (\$/unit, year)
1	0·0003	25	2·000	4000
2	0·0003	20	4·000	2000
3	0·0002	20	6·667	900
4	0·0002	20	20·000	400

Table 1. Data for Silver's examples.

Step 0: We start from an arbitrary $\lambda = 750$.

(*Iteration 1*)
Step 1: Since $f_1(\lambda) = -4{\cdot}099 < 0$, $f_2(\lambda) = 135{\cdot}383 > 0$, so $j = 1$. $T_1(\lambda) = 0{\cdot}196$.
Step 2: Since $g_1(\lambda) = -0{\cdot}008 < 0$, $g_2(\lambda) = 0{\cdot}076 > 0$, so $k = 1$. $T_2(\lambda) = 0{\cdot}206$.
Step 3: Since $T_1(\lambda) < T_2(\lambda)$, decrease λ. $\lambda = \frac{1}{2}(\lambda_h + \lambda_l) = 375$ where $\lambda_h = 750$, $\lambda_l = 0$.

(*Iteration 2*)
Step 1: Since $f_2(\lambda) = -30{\cdot}092 < 0$, $f_3(\lambda) = 49{\cdot}174 > 0$, so $j = 2$. $T_1(\lambda) = 0{\cdot}296$.
Step 2: Since $g_1(\lambda) = -0{\cdot}004 < 0$, $g_2(\lambda) = 0{\cdot}039 > 0$, so $k = 1$. $T_2(\lambda) = 0{\cdot}102$.
Step 3: Since $T_1(\lambda) > T_2(\lambda)$, increase λ. $\lambda = \frac{1}{2}(\lambda_h + \lambda_l) = 562{\cdot}5$ where $\lambda_h = 750$, $\lambda_l = 375$.

(*Iteration 3*)
Step 1: Since $f_1(\lambda) = -39{\cdot}634 < 0$, $f_2(\lambda) = 38{\cdot}825 > 0$, so $j = 1$. $T_1(\lambda) = 0{\cdot}223$.
Step 2: Since $g_1(\lambda) = -\ 0{\cdot}006 < 0$, $g_2(\lambda) = 0{\cdot}057 > 0$, so $k = 1$. $T_2(\lambda) = 0{\cdot}154$.
Step 3: Since $T_1(\lambda) > T_2(\lambda)$, increase λ. $\lambda = \frac{1}{2}(\lambda_h + \lambda_l) = 656{\cdot}25$ where $\lambda h = 750$, $\lambda_l = 562{\cdot}5$.

(*Iteration 4*)
Step 1: Since $f_1(\lambda) = -23{\cdot}143 < 0$, $f_2(\lambda) = 83{\cdot}649 > 0$, so $j = 1$. $T_1(\lambda) = 0{\cdot}211$.
Step 2: Since $g_1(\lambda) = -0{\cdot}007 < 0$, $g_2(\lambda) = 0{\cdot}067 > 0$, so $k = 1$. $T_2(\lambda) = 0{\cdot}180$.
Step 3: Since $T_1(\lambda) > T_2(\lambda)$, increase λ. $\lambda = \frac{1}{2}(\lambda_h + \lambda_l) = 703{\cdot}13$ where $\lambda_h = 750$, $\lambda_l = 656{\cdot}25$.

(*Iteration 5*)
Step 1: Since $f_1(\lambda) = -13{\cdot}940 < 0$, $f_2(\lambda) = 108{\cdot}652 > 0$, so $j = 1$. $T_1(\lambda) = 0{\cdot}204$.
Step 2: Since $g_1(\lambda) = -0{\cdot}008 < 0$, $g_2(\lambda) = 0{\cdot}071 > 0$, so $k = 1$. $T_2(\lambda) = 0{\cdot}193$.
Step 3: Since $T_1(\lambda) > T_2(\lambda)$, increase λ. $\lambda = \frac{1}{2}(\lambda_h + \lambda_l) = 726{\cdot}56$ where $\lambda_h = 750$, $\lambda_l = 703{\cdot}13$.

(*Iteration 6*)
Step 1: Since $f_1(\lambda) = -9{\cdot}099 < 0$, $f_2(\lambda) = 121{\cdot}802 > 0$, so $j = 1$. $T_1(\lambda) = 0{\cdot}200$.
Step 2: Since $g_1(\lambda) = -0{\cdot}008 < 0$, $g_2(\lambda) = 0{\cdot}074 > 0$, so $k = 1$. $T_2(\lambda) = 0{\cdot}200$.
Step 3: Since $T_1(\lambda) = T_2(\lambda)$, $T^* = 0{\cdot}200$ is an optimal common cycle length and $\lambda^* = 726{\cdot}56$.
$\mathrm{x} = (0{\cdot}018, 0, 0, 0)$, $\mathrm{t} = (0{\cdot}091, 0{\cdot}050, 0{\cdot}030, 0{\cdot}010)$

Table 2. Computational procedure for problem 1.

	Schedule 1 (Moon *et al.*)	Schedule 2 (Silver)
Problem 1	3·77%	1·91%
Problem 2	23·57%	14·70%

Table 3. Comparison of savings of average cost for Silver's examples.

Product number	s_i (day)	A_i (\$)	p_i (units/day)	h_i (\$/unit, day)
1	0·5	130	15·2941	0·20896
2	0·75	200	23·5294	0·03188
3	0·5	110	100	0·02321
4	0·125	10	18·75	0·01667
5	0·25	30	47·5	0·01063
6	0·125	20	80	0·00490
7	1	310	400	0·00375
8	0·25	50	300	0·00223
9	0·125	5	150	0·00170
10	0·125	15	300	0·00027

Table 4. Bomberger's data with base demand rate.

Multiples of base demand rates	Load	Schedule 1 (Moon *et al.*)	Schedule 2 (Silver)	(Inman & Jones)
1/2	0·11	9·16*	10·04	9·16
1	0·22	13·26	15·22	13·72
2	0·44	20·52	24·11	20·31

*: $/day.

Table 5. Comparison of average costs for Bomberger's problem.

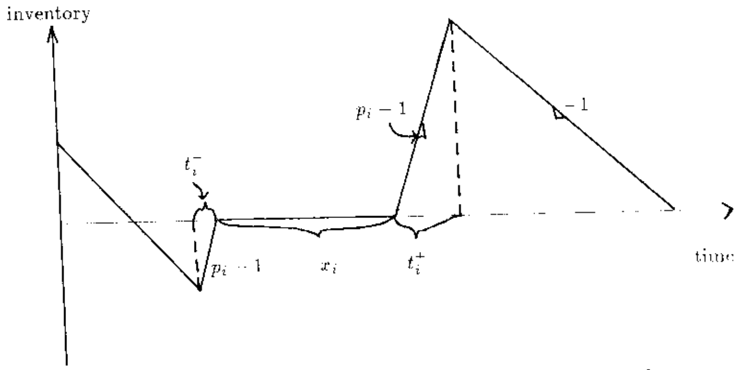

Figure 3. Composition of production times in the backorder case.

5. Extension to the backorder case

We generalize the results to problems where backorders are allowed. Let b_i be the backorder cost per unit time. Divide t_i into t_i^- and t_i^+ where t_i^- is the production time to cover the backorders and t_i^+ is the production time to build-up inventories. As before, x_i denotes the length of time in which we produce to meet demand exactly. See Fig. 3.

The holding cost per cycle for item i is $\frac{1}{2}h_i p_i(p_i-1)(t_i^+)^2$ and the backorder cost per cycle for item i is $\frac{1}{2}b_i p_i(p_i-1)(t_i^-)^2$. From a result in Gallego (1990), optimal $t_i^- = (h_i/h_i+b_i)t_i^*$ and optimal $t_i^+ = (b_i/h_i+b_i)t_i^*$. Consequently, the holding and back-order cost per cycle for item i becomes

$$\frac{1}{2}\frac{h_i b_i}{h_i+b_i}p_i(p_i-1)t_i^2 = \frac{1}{2}\frac{h_i b_i}{h_i+b_i}\bar{r}_i(T-x_i)^2.$$

To solve the backorder case simply apply the algorithm with $h_i \equiv \dfrac{h_i b_i}{h_i+b_i}$.

6. Conclusions

We developed a procedure to solve the Economic Lot Scheduling Problem with the Common Cycle approach where production rates can be controlled during the production runs. This is an important extension over the earlier work where it was assumed that a fixed production rate was established for each item at the beginning of its production. Substantial savings are possible, specially in under-utilized facilities and/or in facilities with large setup costs.

Acknowledgments

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Appendix A: Proof of Theorem 1

Let $z(\mathrm{x}, T)=1/T\Sigma_{i=1}^{m}[A_i+\frac{1}{2}h_i\bar{r}_i(T-x_i)^2]$ and $D=\{(\mathrm{x}, T)\colon\ \Sigma_i s_i+\Sigma_i\bar{r}_i x_i-\kappa T\leqslant 0,\ x_i\geqslant 0,\ i=1,\dots,m\}$. It is clear that $z(\mathrm{x}, T)$ is continuous and that D is closed, hence all level sets of $z(\mathrm{x}, T)$ are closed. The existence of a global minimum for $z(\mathrm{x}, T)$ follows directly once we establish that the level sets are bounded and hence compact. Boundedness is equivalent to the following condition. See Ortega and Rheinboldt (1970).

$$\lim_{k\to\infty} z(\mathrm{x}^k, T^k)=\infty \text{ whenever } \lim_{k\to\infty}\|(\mathrm{x}^k, T^k)\|=\infty\ \forall(\mathrm{x}^k, T^k)\in D. \tag{A 1}$$

Since $z(\mathrm{x}, T)\geqslant\Sigma_i\frac{1}{2}h_i\bar{r}_i(T-2x_i)$, it is sufficient to prove that $\Sigma_i\frac{1}{2}h_i\bar{r}_i(T^k-2x_i^k)$ satisfy the above condition. We use the L_1 norm to prove our result. That is, $\|\mathrm{x}, T\|\geqslant t$ is equivalent to $\Sigma_i x_i+T\geqslant t$. Consider the following primal and dual problem for Linear Programming. We will show that $h(t)\to\infty$ as $t\to\infty$.

(*Primal*)

$$h(t)=\min\sum_i\tfrac{1}{2}h_i\bar{r}_i(T-2x_i)$$

$$\text{s.t.}\quad T+\sum_i x_i\geqslant t$$

$$\kappa T-\sum_i\bar{r}_i x_i\geqslant\sum_i s_i$$

$$x_i\geqslant 0,\ i=1,\dots,m$$

(*Dual 1*)

$$\max\ t\mu+\left(\sum_i s_i\right)v$$

$$\text{s.t.}\quad \kappa v+\mu=\tfrac{1}{2}\sum_i h_i\bar{r}_i \tag{A 2}$$

$$-\bar{r}_i v+\mu\leqslant -h_i\bar{r}_i\qquad i=1,\dots,m \tag{A 3}$$

$$v\geqslant 0,\ \mu\geqslant 0$$

By substituting $\mu=\frac{1}{2}\Sigma_i h_i\bar{r}_i-\kappa v$ from Equation (A 2) into objective function and Equation (A 3), we obtain

(*Dual 2*)

$$\max\left(\sum_i s_i\right)v+\left(\tfrac{1}{2}\sum_i h_i\bar{r}_i-\kappa v\right)t$$

$$\text{s.t.}\qquad \frac{h_i\bar{r}_i+\frac{1}{2}\Sigma_j h_j\bar{r}_j}{(\bar{r}_i+\kappa)}\leqslant v\leqslant\frac{\frac{1}{2}\Sigma_j h_j\bar{r}_j}{\kappa}\qquad i=1,\dots,m$$

Dual 2 is feasible iff $h_i\kappa \leqslant \Sigma_j h_j \bar{r}_j/2$ for all i. Recalling that $h_1 \geqslant h_2 \ldots \geqslant h_m$, feasibility reduces to $h_1\kappa \leqslant \Sigma_j h_j \bar{r}_j/2$, which is guaranteed by condition (f). Clearly, $v^* = \max_i \{[h_i\bar{r}_i + \frac{1}{2}\Sigma_i h_i \bar{r}_i]/[\bar{r}_i+\kappa]\}$ for large t, so by the weak duality theorem

$$h(t) \geqslant \left(\sum_i s_i\right) v^* + \left(\tfrac{1}{2}\sum_i h_i \bar{r}_i - \kappa v^*\right) t.$$

Consequently, $h(t)\to\infty$ as $t\to\infty$ satisfying (A 1). Uniqueness follows from strict convexity of $\kappa z(x, T)$.

Appendix B: Characteristics of f(T, λ) and g(T, λ).

Let $f_0(\lambda) = f(0,\lambda) = -\Sigma_i A_i - \lambda \Sigma_i s_i < 0$ and $f_i(\lambda) = f(\lambda/h_i, \lambda)$, $i = 1, \ldots, m$. For $\lambda \geqslant 0$, we have $f_0(\lambda) \leqslant f_1(\lambda) \leqslant \ldots \leqslant f_m(\lambda)$. Let $\lambda_0 = \infty$ and for $i = 1, \ldots, m$, let λ_i be the positive root of $f_i(\lambda)$. Clearly, $\lambda_0 > \lambda_1 \geqslant \ldots \geqslant \lambda_m$. See Fig. 4.

For $\lambda \in [\lambda_{j+1}, \lambda_j]$, $f_j(\lambda) \leqslant 0 \leqslant f_{j+1}(\lambda)$, and since $f(T,\lambda)$ is increasing in T, $T \in [\lambda/h_j, \lambda/h_{j+1}]$, so it follows that $f_j(\lambda) \leqslant f(T,\lambda) \leqslant f_{j+1}(\lambda)$. Then, by continuity, there exists a unique root $T_1(\lambda) \in [\lambda/h_j, \lambda/h_{j+1}]$ such that $f(T_1(\lambda),\lambda) = 0$. Moreover, it is clear that $T_1(\lambda_j) = \lambda_j/h_j$, $j = 1, 2, \ldots$ m. In fact, for $\lambda \in (\lambda_{j+1}, \lambda_j)$, a little algebra reveals that

$$T_1(\lambda) = \left(\left[\sum_i A_i + \lambda \sum_i s_i - \tfrac{1}{2}\sum_{i\leqq j} \frac{\bar{r}_i}{h_i}\lambda^2\right] \Big/ \tfrac{1}{2}\sum_{i>j} h_i \bar{r}_i\right)^{1/2}.$$

Now fix $\hat{\lambda} \in [\lambda_{j+1}, \lambda_j]$ and $T_1(\hat{\lambda})$ and consider the function

$$f(T_1(\hat{\lambda}),\lambda) = \tfrac{1}{2}\sum_{i\leqq j}\frac{\bar{r}_i}{h_i}\lambda^2 + \tfrac{1}{2}\sum_{i>j} h_i \bar{r}_i T_1(\hat{\lambda})^2 - \sum_i A_i - \lambda \sum_i s_i.$$

Clearly $\hat{\lambda}$ is the largest real root of $f(T_1(\hat{\lambda}),\lambda)$. Then, since $f(T_1(\hat{\lambda}),\lambda)$ is quadratic and convex in λ, it follows that

$$\left.\frac{\partial f(T_1(\hat{\lambda}),\lambda)}{\partial \lambda}\right|_{\lambda=\hat{\lambda}} > 0,$$

i.e.

$$\sum_i s_i - \sum_{i\leqslant j}\frac{\bar{r}_i}{h_i}\hat{\lambda} < 0. \qquad \text{(B 1)}$$

But, this is true for all $\hat{\lambda} \in [\lambda_{j+1}, \lambda_j]$.

Now, from Equation (B 1)

$$\frac{\partial T_1(\lambda)}{\partial \lambda} = -\frac{\partial f(T_1,\lambda)}{\partial \lambda}\Big/\frac{\partial f(T_1,\lambda)}{\partial T_1} = \left(\sum_i s_i - \sum_{i\leqslant j}\frac{\bar{r}_i}{h_i}\lambda\right)\Big/\sum_{i>j} h_i \bar{r}_i T_1 < 0.$$

Thus, $T_1(\lambda)$ is decreasing in $\lambda \in [\lambda_{j+1}, \lambda_j]$, for each $j = 0, 1, \ldots, m-1$, and hence in the interval $[\lambda_m, \lambda_0]$.

Let $g_i(\lambda) = g(\lambda/h_i, \lambda)$, $i = 1, \ldots, m$ and set $g_0(\lambda) = -\infty$. For $\lambda \geqslant 0$, we have $g_0(\lambda) < g_1(\lambda) \leqslant \ldots \leqslant g_m(\lambda)$. $g_i(\lambda)$ is linear in λ. The slope of λ can be either negative or nonnegative. Let $\lambda^0 = 0$ and for $i = 1, \ldots, m$, let λ^i be the positive root of $g_i(\lambda)$ if it exists or ∞. Clearly, $\lambda^0 \leqslant \lambda^1 \leqslant \ldots \leqslant \lambda^m$. Moreover, λ^1 is finite since the slope is negative $(= -\kappa/h_1)$. See Figure 5.

For $\lambda \in [\lambda^k, \lambda^{k+1}]$ with $\lambda^k < \infty$, $k \geqslant 1$, $g_k(\lambda) \leqslant 0 \leqslant g_{k+1}(\lambda)$, and since $g(T,\lambda)$ is increasing in T, $T \in [\lambda/h_k, \lambda/h_{k+1}]$, so it follows that $g_k(\lambda) \leqslant g(T,\lambda) \leqslant g_{k+1}(\lambda)$. It follows that there

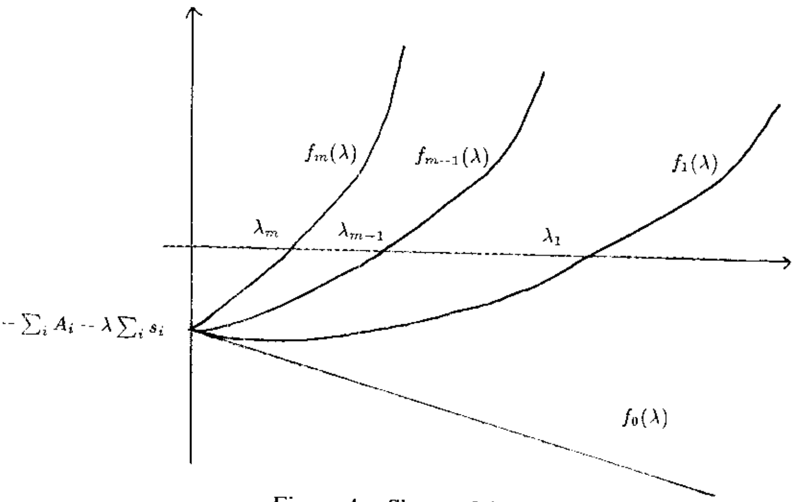

Figure 4. Shape of $f_i(\lambda)$.

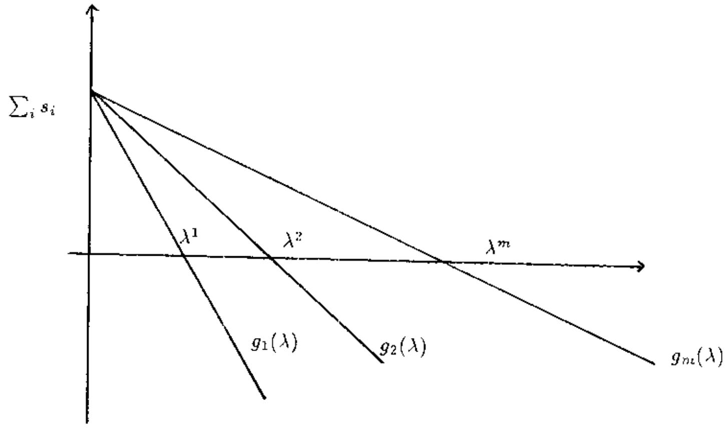

Figure 5. Shape of $g_i(\lambda)$.

exists a unique root $T_2(\lambda) \in [\lambda/h_k, \lambda/h_{k+1}]$ such that $g(T_2(\lambda), \lambda) = 0$. Moreover, it is clear that $T_2(\lambda^k) = \lambda^k/h_k$, $k = 1, 2, \ldots, m$. For $\lambda \in (\lambda^k, \lambda^{k+1})$, a little algebra reveals that

$$T_2(\lambda) = \left[\lambda \sum_{i \leq k} \frac{\bar{r}_i}{h_i} - \sum_i s_i \right] \Bigg/ \left[\sum_{i \leq k} \bar{r}_i - \kappa \right].$$

Consequently

$$\frac{\partial T_2(\lambda)}{\partial \lambda} = \sum_{i \leq k} \frac{\bar{r}_i}{h_i} \Bigg/ \left[\sum_{i \leq k} \bar{r}_i - \kappa \right] > 0.$$

Thus, $T_2(\lambda)$ is piecewise linear and increasing in $\lambda \in [\lambda^k, \lambda^{k+1}]$, and hence in the interval $[\lambda^0, \lambda^m]$.

Since $T_1(\lambda)$ (respectively, $T_2(\lambda)$) is strictly decreasing (respectively increasing), there exists a unique λ^* such that $T_1(\lambda^*) = T_2(\lambda^*)$.

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